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McKay Quivers and Terminal Quotient Singularities in Dimension 3

by

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Contents

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Declarations

Section 2 repeats standard results from the literature for the convenience of the reader. Also Section 3.1 is mostly reminders of standard facts. Apart from that, I declare that the work contained in this thesis is original except where otherwise stated in the text. I confirm that this thesis has not been submitted anywhere else for any degree.

Abstract

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. For such G, the quotient variety $X = \mathbb{C}^3/G$ is not Gorenstein and has a terminal singularity. The singular variety X has the economic resolution which is "close to being crepant". In this paper, we prove that the economic resolution of the quotient variety $X = \mathbb{C}^3/G$ is isomorphic to the birational component of a moduli space of θ-stable McKay quiver representations for a suitable GIT parameter θ. Moreover, we conjecture the moduli space of θ-stable McKay quiver representations is irreducible, and prove this for $a = 2$ and in a number of special examples.

Chapter 1

Introduction

The motivation of this work stems from the philosophy of the *McKay correspondence*, which says that if a finite group G acts on a variety M, then the crepant resolutions of the quotient variety M/G have information of the G-equivariant geometry of M [29].

Let G be a finite subgroup of $\mathrm{GL}_n(\mathbb{C})$. A G-equivariant coherent sheaf $\mathcal{F}$ on $\mathbb{C}^n$ is called a *G-constellation* if its global sections $\mathrm{H}^0(\mathcal{F})$ are isomorphic to the regular representation $\mathbb{C}[G]$ of G as a G-module. In particular, the structure sheaf of a G-invariant subscheme $Z \subset \mathbb{C}^n$ with $\mathrm{H}^0(\mathcal{O}_Z)$ isomorphic to $\mathbb{C}[G]$ as a G-module, which is called a *G-cluster*, is a G-constellation. It is known that G-clusters are θ-stable G-constellations for a particular choice of GIT stability parameter θ [13].

For a finite group $G \subset \mathrm{SL}_2(\mathbb{C})$, Ito and Nakamura [14] introduced $G\text{-Hilb}\,\mathbb{C}^2$ which is the fine moduli space parametrising G-clusters and proved that $G\text{-Hilb}\,\mathbb{C}^2$ is the minimal resolution of $\mathbb{C}^2/G$. In the celebrated paper [1], Bridgeland, King and Reid proved that for a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$, $G\text{-Hilb}\,\mathbb{C}^3$ is a crepant resolution of the quotient variety $\mathbb{C}^3/G$. Also Craw and Ishii [2] showed that in the case of a finite abelian group $G \subset \mathrm{SL}_3(\mathbb{C})$, *any* projective crepant resolution can be realised as the fine moduli space of θ-stable G-constellations for a suitable stability parameter θ.

For a finite abelian group $G \subset \mathrm{GL}_n(\mathbb{C})$ and a generic GIT parameter $\theta \in \Theta$, Craw, Maclagan and Thomas [4] showed that the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations has a unique irreducible component Y_θ which contains the torus $T := (\mathbb{C}^\times)^n/G$. So the irreducible component is birational to the quotient variety $\mathbb{C}^n/G$. The component Y_θ is called the *birational component*[1] of $\mathcal{M}_\theta$.

On the other hand, it is shown [23, 28] that a 3-fold cyclic quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if and only if G is of type $\frac{1}{r}(1, a, r-a)$

[1] This component is also called the coherent component.

with a coprime to r. In this case, X have a preferred toric resolution, called the *economic resolution.* For the group G of type $\frac{1}{r}(1, a, r-a)$, G-Hilb $\mathbb{C}^3$ is smooth and isomorphic to the economic resolution of X if and only if $a = 1$ or $r - 1$ as shown in [17]. Kędzierski [16] proved that there exists a Weyl chamber $\mathfrak{C}$ in Θ such that the normalization of the birational component Y_θ of the moduli space of θ-stable G-constellations is isomorphic to the economic resolution Y of $X = \mathbb{C}^3/G$. To show this, he found a suitable family over the economic resolution Y and a chamber $\mathfrak{C}$ such that G-constellations in the family are θ-stable for $\theta \in \mathfrak{C}$. His original description of stability parameters is a set of inequalities, but one can show that his stability parameters form an open Weyl chamber and this is easy to describe using the A_{r-1} root system.

Main results

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r, i.e. G is the subgroup generated by the diagonal matrix $\mathrm{diag}(\epsilon, \epsilon^a, \epsilon^{r-a})$ where ϵ is a primitive rth root of unity. The quotient variety $X = \mathbb{C}^3/G$ is not Gorenstein and has terminal singularities. Moreover, the singular variety $X = \mathbb{C}^3/G$ has no crepant resolution. However, there exist economic resolutions which are close to being crepant (see Section 5.7 in [28]). The economic resolution can be obtained by a toric method, which is called weighted blowups.

In this paper, we prove that the economic resolution Y is isomorphic to an irreducible component of the moduli space of G-equivariant sheaves on $\mathbb{C}^3$. More precisely, we have the following theorem.

Theorem 1.0.1 (Corollary 4.3.2)**.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. The economic resolution Y of $X = \mathbb{C}^3/G$ is isomorphic to the birational component Y_θ of the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations for a suitable parameter θ.*

To prove this, we introduce *generalized G-graphs* and *round down functions.* A *generalized* G-graph Γ is a generalized version of Nakamura G-graph in [25]. A G-graph corresponds to a torus invariant G-constellation. We define a toric affine open set $U(\Gamma)$ associated to a G-graph Γ and a family of G-constellations over $U(\Gamma)$. These give us a local chart of the moduli space of θ-stable McKay quiver representations for suitable parameter θ. On the other hand, the round down functions are related to weighted blowups. For each step of the weighted blowups, we define three *round down functions*, that are maps between monomial lattices. The round down functions are used for finding admissible G-graphs, which define the universal family over the economic resolution Y.

Moreover, we prove that our stability parameters form an open Weyl chamber, which coincides with the chamber in [16]. With Section 5.1, we can see that the chamber is a full chamber in the GIT stability parameter space.

Layout of this thesis

In Chapter 2, we define *(generalized) G-graphs* and we review standard results on moduli spaces of G-constellations. Using certain G-graphs, we describe the birational component of the moduli space of θ-stable G-constellations. Chapter 3 explains how to obtain the economic resolutions using toric methods and defining round down functions. The round down functions will play a big role in finding admissible G-graphs. Chapter 4 contains our main theorem. In Section 4.1, we explain the way to find a set of admissible G-graphs in a recursive way using round down functions. In Section 4.2, we prove that the admissible G-graphs in Section 4.1 are θ-stable for parameters θ in a suitable chamber. Moreover, we prove that the GIT parameters form an open Weyl chamber. In Section 4.3, we state the main theorem and conjectures. Chapter 5 contains further results. Section 5.1 presents a description of θ-stable torus invariant A-constellations for A of type $\frac{1}{r}(1, r-1)$ if θ is in an open Weyl chamber of A_{r-1}. Section 5.2 investigates the chamber structure of GIT stability parameters. Section 5.3 proves that the moduli space of θ-stable McKay quiver representations is irreducible if $a = 2$.

Chapter 2

G-graphs and G-constellations

This section introduces a (generalized) G-graph which is a generalized version of Nakamura G-graphs in [25]. As Nakamura G-graphs are associated with torus invariant G-clusters, our G-graphs are associated with torus invariant G-constellations. If a G-graph Γ satisfies a certain condition, then we call the G-graph a *G-iraffe*. For each G-iraffe Γ, we define a toric affine open set $U(\Gamma)$ and a family over the open set $U(\Gamma)$.

In this section, we restrict ourselves to the case where a group G is a finite cyclic group in $\mathrm{GL}_3(\mathbb{C})$. It is possible to generalize part of the argument to include general small abelian groups in $\mathrm{GL}_n(\mathbb{C})$ for any dimension n. However, we prefer to focus on this case where we can avoid the difficulty of notation.

2.1 Moduli of quiver representations

In this section, we briefly review the construction of moduli spaces of quiver representations introduced in [18].

2.1.1 Quivers and their representations

A *quiver* Q is a directed graph with a set of vertices $I = Q_0$ and a set of arrows Q_1. For an arrow $a \in Q_1$, let $\mathrm{h}(a)$ (resp. $\mathrm{t}(a)$) denote the head (resp. tail) of the arrow a:

$$\mathrm{t}(a) \xrightarrow{a} \mathrm{h}(a).$$

One can define the *path algebra* of a quiver Q to be the $\mathbb{C}$-algebra whose basis is nontrivial paths in Q and trivial paths corresponding to the vertices of Q and whose multiplication is given by the concatenation of two paths.

A *representation* of a quiver Q is a collection of $\mathbb{C}$-vector spaces V_i for each vertex $i \in I$ and linear maps $V_i \to V_j$ for each arrow from i to j. For a representation V, the I-tuple $(\dim_{\mathbb{C}} V_i)_{i\in I} \in \mathbb{Z}^I_{\geq 0}$ is called the *dimension vector* of V denoted by $\underline{\dim}\,(V)$. A representation (U, ξ') of a quiver Q is called a *subrepresentation* of a representation (V, ξ) if U is an I-graded subspace of V such that $\xi_a(U_{\mathrm{t}(a)}) \subset U_{\mathrm{h}(a)}$ for all $a \in Q_1$ and ξ' is the restriction of ξ to U.

It is well known that the abelian category of representations of a quiver Q is equivalent to the category of finitely generated left modules of the path algebra of Q.

Let us fix a dimension vector $\mathbf{v} = (v_i)_{i\in I}$. Let $\mathrm{Rep}(Q, \mathbf{v})$ denote the representation space of Q with dimension vector $\mathbf{v}$:

$$\mathrm{Rep}(Q, \mathbf{v}) = \bigoplus_{a\in Q_1} \mathrm{Hom}(V_{\mathrm{t}(a)}, V_{\mathrm{h}(a)}) = \bigoplus_{a:i\to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}),$$

which is an affine space. Note that the reductive group $\mathrm{GL}(\mathbf{v}) := \prod_{i\in I} \mathrm{GL}_{v_i}$ acts on $\mathrm{Rep}(Q, \mathbf{v})$ as basis change.

One can see that

$$\mathrm{Rep}(Q, \mathbf{v}) \longrightarrow \mathrm{Rep}(Q, \mathbf{v}) /\!\!/ \mathrm{GL}(\mathbf{v}) := \mathrm{Spec}\,\mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{\mathrm{GL}(\mathbf{v})}$$

is a categorical quotient and that $\mathrm{Rep}(Q, \mathbf{v}) /\!\!/ \mathrm{GL}(\mathbf{v})$ is an affine variety.

Remark 2.1.1. Geometric points of $\mathrm{Rep}(Q, \mathbf{v}) /\!\!/ \mathrm{GL}(\mathbf{v})$ correspond to $\mathrm{GL}(\mathbf{v})$-orbits of semisimple representations of Q whose dimension is $\mathbf{v}$. ♦

2.1.2 Background: Geometric Invariant Theory

In this section, we present results from standard Geometric Invariant Theory (GIT), cf. [22].

Definition 2.1.2. Let G be a reductive group acting on an affine variety X. A surjective morphism $\psi\colon X \to Y$ is a *good quotient* if:

(i) ψ is constant on G-orbits.

(ii) the natural map $\mathcal{O}_Y(U) \to \psi_*\mathcal{O}_X(U)$ induces $\mathcal{O}_Y(U) = (\psi_*\mathcal{O}_X)^G(U)$ for any open set $U \subset Y$.

(iii) $\psi(W)$ is closed in Y for any G-invariant closed set $W \subset X$.

(iv) $\psi(W_1) \cap \psi(W_2) = \emptyset$ for two disjoint G-invariant closed sets W_1, W_2 of X.

Moreover, if Y is an orbit space, then $\psi\colon X \to Y$ is called a *geometric quotient.*

Consider an affine algebraic variety X with a reductive group G acting on it. Given a character $\chi\colon G \to \mathbb{C}^\times$, $f \in \mathbb{C}[X]$ is called a χ semi-invariant function if

$$f(g \cdot x) = \chi(g) f(x) \quad x \in X,\ \forall g \in G.$$

Let $\mathbb{C}[X]_{\chi^n}$ denote the $\mathbb{C}$-vector space of all χ^n semi-invariant functions. One defines the *semistable locus* as

$$X^{ss}(\chi) := \left\{ x \in X \,\middle|\, \exists n \geq 1, f \in \mathbb{C}[X]_{\chi^n} \text{ such that } f(x) \neq 0 \right\}$$

and the *stable locus* as

$$X^{s}(\chi) := \left\{ x \in X^{ss}(\chi) \,\middle|\, G \cdot x \text{ is closed in } X^{ss}(\chi), \text{ the stabiliser } G_x \text{ is finite} \right\}.$$

The quasiprojective variety

$$X /\!\!/_{\chi} G := \operatorname{Proj} \Big(\bigoplus_{n \geq 0} \mathbb{C}[X]_{\chi^n} \Big)$$

is called a *GIT quotient* corresponding to χ. In particular, if the character $\chi = 0$, i.e. θ is trivial, then $\mathbb{C}[X]_{\chi^n} = \mathbb{C}[X]^G$ for all n so we have

$$X /\!\!/_{0} G = \operatorname{Spec} \mathbb{C}[X]^G$$

which is an affine variety. Thus we have a canonical projective morphism

$$X /\!\!/_{\chi} G \to \operatorname{Spec} \mathbb{C}[X]^G.$$

Remark 2.1.3. Let G be a reductive group acting on an affine variety X. Fix a character χ of G. For each positive integer d, define the dth *Veronese subalgebra* of $\bigoplus_{n \geq 0} \mathbb{C}[X]_{\chi^n}$ to be

$$\bigoplus_{n \geq 0} \mathbb{C}[X]_{\chi^{dn}}.$$

One can show that the inclusion of the subalgebra induces an isomorphism of algebraic varieties

$$X /\!\!/_{\chi} G \xrightarrow{\sim} X /\!\!/_{\chi^d} G.$$

Thus any positive multiple of a character χ gives the same GIT quotient as χ. ♦

As is well known by GIT [22], the quasiprojective variety $X /\!\!/_{\chi} G$ is a categorical quotient $X^{ss}(\chi)$ by G.

Theorem 2.1.4 (Geometric Invariant Theory [22])**.** *Let G be a reductive group acting on an affine variety X and χ a character of G. Then:*

(i) $\pi\colon X^{ss}(\chi) \to X /\!\!/_\chi G$ *is a good quotient of* $X^{ss}(\chi)$ *by* G.

(ii) there exists an open subset Y of $X /\!\!/_\chi G$ such that Y is a geometric quotient of $X^s(\chi)$ by G, i.e. an orbit space.

(iii) the GIT quotient $X /\!\!/_\chi G$ is projective over the affine variety $\operatorname{Spec} \mathbb{C}[X]^G$.

Remark 2.1.5. Assume that $X^s(\chi) = X^{ss}(\chi)$. Let $\pi\colon X /\!\!/_\chi G \to X^s(\chi)/G$ be the GIT quotient. Then π is a geometric quotient. Let U be a G-invariant affine open set in $X^{ss}(\chi)$. Then

$$\pi|_U\colon U \to \pi(U)$$

is a good quotient and $\pi(U) = \operatorname{Spec} \mathbb{C}[U]^G$ is an open set of $X^s(\chi)/G$. ♦

The following theorem is helpful to understand the local behaviour of the GIT quotients.

Theorem 2.1.6 (Luna's Étale Slice Theorem [12, 21])**.** *Let G be a reductive group acting on an affine variety X. Assume that $\pi\colon X \to X /\!\!/ G$ is a good quotient. Let $x \in X$ be a point with closed G-orbit $G \cdot x$. Then there exists a G_x-invariant locally closed affine subset V of X containing x such that the G-action on X induces an étale G-equivariant morphism $\psi\colon G \times_{G_x} V \to X$. Moreover, ψ induces an étale morphism $V /\!\!/ G_x \to X /\!\!/ G$, and the following diagram*

$$\begin{array}{ccc} G \times_{G_x} V & \to & X \\ \downarrow & & \downarrow \\ V /\!\!/ G_x & \to & X /\!\!/ G \end{array}$$

is Cartesian.

2.1.3 Moduli spaces of quiver representations

This section explains a notion of stability on quiver representations introduced by King [18]. His main result is that the notion of stability on quiver representations and the notion of GIT stability are equivalent and that we can construct a fine moduli space of quiver representations in a certain case.

An element $\theta \in \mathbb{Q}^I$ can be thought as a group homomorphism from the Grothendieck group of representations of Q to $\mathbb{Q}$ defined by

$$\theta(V) := \sum_{i \in I} \theta_i \dim_C V_i = \theta \cdot \mathbf{v}$$

where V is a representation of Q with dimension vector $\mathbf{v}$.

Definition 2.1.7. Let V be a $\mathbf{v}$-dimensional representation of a quiver Q. For a parameter $\theta \in \mathbb{Q}^I$ satisfying $\theta \cdot \mathbf{v} = 0$, we say that:

(i) V is *θ-semistable* if $\theta(W) \geq 0$ for any subrepresentation W of V.

(ii) V is *θ-stable* if $\theta(W) > 0$ for any nonzero proper subrepresentation W of V.

(iii) θ is *generic* if every θ-semistable representation is θ-stable.

The parameter $\theta \in \mathbb{Q}^I$ plays the same role as χ does in Section 2.1.2. The character χ_θ defined by

$$\chi_\theta(g) := \prod_{i \in I} \det(g_i)^{\theta_i}$$

for $g = (g_i) \in \mathrm{GL}(\mathbf{v})$ vanishes on the diagonal matrices $\mathbb{C}^\times \in \mathrm{GL}(\mathbf{v})$ if and only if $\theta \cdot \mathbf{v} = 0$.

King [18] shows that a representation $V \in \mathrm{Rep}(Q, \mathbf{v})$ is θ-semistable (resp. θ-stable) if and only if the corresponding point $V \in \mathrm{Rep}(Q, \mathbf{v})$ is χ_θ-semistable (resp. χ_θ-stable). Moreover:

Theorem 2.1.8 (King [18])**.** *Let $\mathbf{v}$ be a dimension vector. Assume a parameter $\theta \in \mathbb{Q}^I$ satisfies $\theta \cdot \mathbf{v} = 0$.*

(i) The quasiprojective variety

$$\mathcal{M}_\theta(Q, \mathbf{v}) := \mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]_{\chi_\theta^n}\right)$$

is a coarse moduli space of θ-semistable $\mathbf{v}$-dimensional representations of Q up to S-equivalence.

(ii) If θ is generic, $\mathcal{M}_\theta(Q, \mathbf{v})$ is a fine moduli space of θ-stable $\mathbf{v}$-dimensional representations of Q.

(iii) The variety $\mathcal{M}_\theta(Q, \mathbf{v})$ is projective over $\mathrm{Spec}\, \mathbb{C}[\mathrm{Rep}(Q, \mathbf{v})]^{\mathrm{GL}(\mathbf{v})}$.

Remark 2.1.9. By Luna's Étale Slice Theorem, if θ is generic, then the quotient map

$$\pi\colon \operatorname{Rep}^s(Q,\mathbf{v}) \to \mathcal{M}_\theta(Q,\mathbf{v})$$

is a principal $\mathrm{GL}(\mathbf{v})/\mathbb{C}^\times$-bundle. ♦

2.2 McKay quiver and G-constellations

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{r}(\alpha_1,\alpha_2,\alpha_3)$. Let ρ_i be the irreducible representation of G whose weight is i. Since G is abelian, every irreducible representation is one-dimensional and the number of irreducible representation is equal to the order of G. We can identify $I := \mathrm{Irr}(G)$ with $\mathbb{Z}/r\mathbb{Z}$. Note that the inclusion $G \subset \mathrm{GL}_3(\mathbb{C})$ induces a natural representation of G on $\mathbb{C}^3$, which can be decomposed as

$$\rho_{\alpha_1} \oplus \rho_{\alpha_2} \oplus \rho_{\alpha_3}.$$

2.2.1 McKay quiver representations

Definition 2.2.1. (McKay quiver) The *McKay quiver* of G is the quiver whose vertex set is the set I of irreducible representations of G and the number of arrows from ρ_i to ρ_j is the dimension of $\mathrm{Hom}_G(\rho_j, (\rho_{\alpha_1} \oplus \rho_{\alpha_2} \oplus \rho_{\alpha_3}) \otimes \rho_i)$.

Since G has r irreducible representations, the McKay quiver of G has r vertices $\rho_0, \dots, \rho_{r-1}$. For two irreducible G-representations ρ_i and ρ_j,

$$\begin{aligned}\mathrm{Hom}_G\left(\rho_j, (\rho_{\alpha_1} \oplus \rho_{\alpha_2} \oplus \rho_{\alpha_3}) \otimes \rho_i\right) &= \mathrm{Hom}_G(\rho_j, \textstyle\bigoplus_{k=1}^{3} \rho_{\alpha_k} \otimes \rho_i) \\ &= \bigoplus_{k=1}^{3} \mathrm{Hom}_G(\rho_j, \rho_{i+\alpha_k}),\end{aligned}$$

and by Schur's lemma

$$\dim \mathrm{Hom}_G(\rho_j, \rho_{i+\alpha_k}) = \begin{cases} 1 & \text{if } j = i + \alpha_k \mod r, \\ 0 & \text{otherwise.}\end{cases}$$

Thus the McKay quiver has $3r$ arrows. Let x_i, y_i, z_i denote the arrow from ρ_i to $\rho_{i+\alpha_1}$, $\rho_{i+\alpha_2}$, $\rho_{i+\alpha_3}$, respectively. We are interested in the McKay quiver with the following commutation relations:

$$\begin{cases} x_i y_{i+\alpha_1} - y_i x_{i+\alpha_2}, \\ x_i z_{i+\alpha_1} - z_i x_{i+\alpha_3}, \\ y_i z_{i+\alpha_2} - z_i y_{i+\alpha_3}. \end{cases} \tag{2.2.2}$$

Definition 2.2.3. A *McKay quiver representation* is a representation of the McKay quiver of dimension $(1, \ldots, 1)$ with the relations (2.2.2), i.e. it is a collection of one-dimensional $\mathbb{C}$-vector spaces V_i for each $\rho_i \in G^\vee$, and a collection of linear maps from V_i to V_j assigned to each arrow from ρ_i to ρ_j which satisfy the commutation relations (2.2.2).

Example 2.2.4. Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite group of type $\frac{1}{12}(1,5,7)$, i.e. $r = 12$ and $a = 5$. The set of irreducible representations of G is $\{\rho_i \,\big|\, 0 \le i \le 11\}$ and the induced representation is isomorphic to $\rho_1 \oplus \rho_5 \oplus \rho_7$. The McKay quiver of G has 12 vertices and 36 arrows.

After fixing basis on vector spaces attached to vertices, the McKay quiver representations are in 1-to-1 correspondence with points of the closed subscheme of the affine space

$$\mathbb{C}^{3r} = \operatorname{Spec} \mathbb{C}[x_0, \ldots, x_{r-1}, y_0, \ldots, y_{r-1}, z_0, \ldots, z_{r-1}]$$

defined by the commutation relations (2.2.2). ♦

Let $\operatorname{Rep} G$ denote the McKay quiver representation space of G. Note that its coordinate ring is

$$\mathbb{C}[\operatorname{Rep} G] = \mathbb{C}\left[x_i, y_i, z_i \,\middle|\, i \in I\right] / I_G$$

where I_G is the ideal generated by the quadrics in (2.2.2).

Let $\delta = (1, \ldots, 1) \in \mathbb{Z}^I_{\ge 0}$. The reductive group $\mathrm{GL}(\delta) := \prod_{i \in I} \mathbb{C}^\times = (\mathbb{C}^\times)^r$ acts on $\operatorname{Rep} G$ by basis change. Note that $\mathrm{GL}(\delta)$-orbits are in 1-to-1 correspondence with isomorphism classes of the McKay quiver representations.

Consider the algebraic torus $\mathbf{T} = (\mathbb{C}^\times)^3$ acting on $\operatorname{Rep} G$ by

$$(t_1, t_2, t_3) \cdot (x_i, y_i, z_i) = (t_1 x_i, t_2 y_i, t_3 z_i).$$

One can see that $\mathbf{T}$-action commutes with $\mathrm{GL}(\delta)$-action. This action naturally comes from the notion of *G-constellations*, which are a certain kind of coherent sheaves on $\mathbb{C}^3$ (see Remark 2.2.15).

We define the GIT parameter space Θ to be

$$\Theta := \left\{\theta \in \mathbb{Q}^I \,\middle|\, \theta \cdot \delta = 0\right\}.$$

By Theorem 2.1.8, we know that:

(i) the quasiprojective scheme

$$\mathcal{M}_\theta := \operatorname{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[\operatorname{Rep} G]_{\chi_\theta^n}\right)$$

is a coarse moduli space of θ-semistable McKay quiver representations up to S-equivalence.

(ii) if θ is generic, $\mathcal{M}_\theta$ is a fine moduli space of θ-stable McKay quiver representations of Q.

(iii) $\mathcal{M}_\theta$ is projective over $\operatorname{Spec}\mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$.

Remark 2.2.5. The affine scheme $\operatorname{Spec}\mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$ contains the quotient variety $\mathbb{C}^3/G$ as a closed subvariety (see Remark A.0.2). ♦

2.2.2 G-constellations

Definition 2.2.6. A *G-constellation* on $\mathbb{C}^3$ is a G-equivariant $\mathbb{C}[x,y,z]$-module $\mathcal{F}$ on $\mathbb{C}^3$, which is isomorphic to the regular representation $\mathbb{C}[G]$ of G as a G-module.

Remark 2.2.7. Any G-constellation $\mathcal{F}$ is isomorphic to $\bigoplus_i \mathbb{C}\rho_i$ as a vector space. ♦

The representation ring $R(G)$ of G is $\bigoplus_{\rho\in G^\vee} \mathbb{Z}\rho$. Define the GIT stability parameter space

$$\begin{aligned}\Theta &= \left\{\theta \in \operatorname{Hom}_{\mathbb{Z}}(R(G),\mathbb{Q}) \,\middle|\, \theta\left(\mathbb{C}[G]\right) = 0\right\}\\ &= \left\{\theta = (\theta^i) \in \mathbb{Q}^r \,\middle|\, \Sigma_{i\in I}\theta^i = 0\right\}.\end{aligned}$$

Definition 2.2.8. For a stability parameter $\theta \in \Theta$, we say that:

(i) a G-constellation $\mathcal{F}$ is *θ-semistable* if $\theta(\mathcal{G}) \geq 0$ for any nonzero proper submodule $\mathcal{G} \subset \mathcal{F}$.

(ii) a G-constellation $\mathcal{F}$ is *θ-stable* if $\theta(\mathcal{G}) > 0$ for any nonzero proper submodule $\mathcal{G} \subset \mathcal{F}$.

(iii) θ is *generic* if every θ-semistable object is θ-stable.

Remark 2.2.9. It is known that the language of G-constellations is the same as the language of the McKay quiver representations. Thus we can construct the moduli spaces of G-constellations by Geometric Invariant Theory as in Section 2.1. ♦

Let $\mathcal{M}_\theta$ denote the moduli space of θ-stable G-constellations. Ito and Nakajima [13] showed that $G\text{-Hilb}\,\mathbb{C}^3$ is isomorphic to $\mathcal{M}_\theta$ if θ is in the following set:

$$\Theta_+ := \left\{\theta \in \Theta \,\middle|\, \theta(\rho) > 0 \text{ for nontrivial } \rho \neq \rho_0\right\}. \tag{2.2.10}$$

Let Z be a G-orbit in the algebraic torus $\mathbf{T} := (\mathbb{C}^\times)^3 \subset \mathbb{C}^3$. Then $\mathrm{H}^0(\mathcal{O}_Z)$ is isomorphic to $\mathbb{C}[G]$, thus it is a G-constellation. Moreover, since Z is a free G-orbit, $\mathcal{O}_Z$ has no nonzero proper submodules. Hence it follows that $\mathcal{O}_Z$ is θ-stable for any parameter θ. Thus for any parameter θ, there exists a natural embedding of the torus $T := (\mathbb{C}^\times)^3/G$ into $\mathcal{M}_\theta$.

Remark 2.2.11. The existence of the natural embedding of the torus $T := (\mathbb{C}^\times)^3/G$ into $\mathcal{M}_\theta$ can be proved by Luna's Étale Slice Theorem as is standard in the theory of moduli spaces of sheaves (e.g. see [12]). ♦

Lemma 2.2.12. *Let Z be a free G-orbit in $\mathbb{C}^3$. Then $\mathcal{O}_Z$ is a G-constellation supported on the free G-orbit Z. Conversely, if a G-constellation $\mathcal{F}$ is supported on a free G-orbit $Z \subset \mathbb{C}^3$, then $\mathcal{F}$ is isomorphic to $\mathcal{O}_Z$ as a G-constellation.*

Proof. For the first statement, one can refer to [24].

To prove the second statement, let $\mathcal{F}$ be a G-constellation whose support is a free G-orbit Z.

Then $\mathcal{F}$ has no nonzero proper submodules. Indeed, for a nonzero submodule $\mathcal{G}$ of $\mathcal{F}$, the support of $\mathcal{G}$ is a G-invariant nonempty subset of the free G-orbit Z. As Z is a free G-orbit, the support of $\mathcal{G}$ is Z. Since $\mathcal{F}_x$ is 1-dimensional for any $x \in Z$, it follows that $\mathcal{G}_x = \mathcal{F}_x$ and hence $\mathcal{G} = \mathcal{F}$.

Consider $\psi\colon \mathbb{C}[x,y,z] \to \mathcal{F}$ defined by $f \mapsto f * e_0$ where e_0 is a basis of $\mathbb{C}\rho_0$. As $\mathcal{F}$ has no nonzero proper submodules, ψ is surjective. Since the support of $\mathcal{F}$ is Z, it follows that I_Z is in the kernel of ψ. Thus we have

$$\mathcal{O}_Z = \mathbb{C}[x,y,z]/I_Z \geq \mathbb{C}[x,y,z]/\ker(\psi) \cong \mathcal{F}.$$

From the fact that both $\mathcal{O}_Z$ and $\mathcal{F}$ are G-constellations, it follows that $\mathcal{O}_Z \cong \mathcal{F}$ as $\dim_{\mathbb{C}} \mathcal{O}_Z = \dim_{\mathbb{C}} \mathcal{F}$. □

Craw, Maclagan and Thomas [4] proved the following theorem.

Theorem 2.2.13 (Craw, Maclagan and Thomas [4])**.** *Let $\theta \in \Theta$ be generic. Then $\mathcal{M}_\theta$ has a unique irreducible component Y_θ which contains the torus $T := (\mathbb{C}^\times)^n/G$. Moreover Y_θ satisfies the following properties:*

(i) Y_θ is a not-necessarily-normal toric variety which is birational to the quotient variety $\mathbb{C}^3/G$.

(ii) Y_θ is projective over the quotient variety $\mathbb{C}^3/G$.

$$\begin{tikzcd} Y_\theta \arrow[r, hook, "\text{irr.}"'] \arrow[d] & \mathcal{M}_\theta \arrow[d] \\ \mathbb{C}^3/G \arrow[r, hook, "\text{closed}"'] & \mathcal{M}_0 \end{tikzcd}$$

Remark 2.2.14. We call the unique irreducible component Y_θ of $\mathcal{M}_\theta$ the *birational component*. For generic $\theta \in \Theta$, Craw, Maclagan and Thomas [4] constructed the birational component Y_θ as GIT quotient of a reduced irreducible affine scheme by an algebraic torus. From this, it follows that Y_θ is irreducible and reduced. ♦

Remark 2.2.15. Since the algebraic torus $\mathbf{T}$ acts on $\mathbb{C}^3$, $\mathbf{T}$ acts on the moduli space $\mathcal{M}_\theta$ naturally. Fixed points of the $\mathbf{T}$-action play a crucial role in the study of the moduli space $\mathcal{M}_\theta$. Note that this $\mathbf{T}$-action is the same as the $\mathbf{T}$-action in Section 2.2.1. ♦

2.3 Abelian group actions and toric geometry

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$, i.e. G is the subgroup generated by the diagonal matrix $\mathrm{diag}(\epsilon^{\alpha_1}, \epsilon^{\alpha_2}, \epsilon^{\alpha_3})$ where ϵ is a primitive rth root of unity. The group G acts naturally on $S := \mathbb{C}[x, y, z]$. Define the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$$

which is an overlattice of $\overline{L} = \mathbb{Z}^3$ of finite index. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$. Set $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$ and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$. The dual lattices $\overline{M}$ and M can be identified with Laurent monomials and G-invariant Laurent monomials, respectively. The embedding of G into the torus $(\mathbb{C}^\times)^3 \subset \mathrm{GL}_3(\mathbb{C})$ induces a surjective homomorphism

$$\mathrm{wt}\colon \overline{M} \longrightarrow G^\vee$$

where $G^\vee := \mathrm{Hom}(G, \mathbb{C}^\times)$ is the character group of G. Note that M is the kernel of the map wt.

Remark 2.3.1. There are two isomorphisms of abelian groups $L/\mathbb{Z}^3 \to G$ and $\overline{M}/M \to G^\vee$. ♦

Let $\overline{M}_{\geq 0}$ denote genuine monomials in $\overline{M}$, i.e.

$$\overline{M}_{\geq 0} = \left\{ x^{m_1} y^{m_2} z^{m_3} \in \overline{M} \,\middle|\, m_1, m_2, m_3 \geq 0 \right\}.$$

For a set $A \subset \mathbb{C}[x^\pm, y^\pm, z^\pm]$, let $\langle A \rangle$ denote the $\mathbb{C}[x, y, z]$-submodule of $\mathbb{C}[x^\pm, y^\pm, z^\pm]$ generated by A.

Let σ_+ be the cone in $L_\mathbb{R} := L \otimes_\mathbb{Z} \mathbb{R}$ generated by e_1, e_2, e_3, i.e.

$$\sigma_+ := \mathrm{Cone}(e_1, e_2, e_3).$$

For the cone σ_+ and the lattice L, we define a corresponding affine toric variety

$$U_{\sigma_+} := \mathrm{Spec}\, \mathbb{C}[\sigma_+^\vee \cap M].$$

Note that U_{σ_+} is the quotient variety $X = \mathbb{C}^3/G = \mathrm{Spec}\, \mathbb{C}[x, y, z]^G$ as M is the G-invariant Laurent monomials.

Remark 2.3.2. As is usual in toric geometry, the affine toric variety of the cone σ_+ with the lattice $\overline{L}$ is

$$\mathbb{C}^3 = \mathrm{Spec}\, \mathbb{C}[x, y, z] = \mathrm{Spec}\, \mathbb{C}[\sigma_+^\vee \cap \overline{M}].$$

The quotient map $\mathbb{C}^3 \to X$ is induced by the inclusion $\overline{L} \subset L$. ♦

Let $\square$ be the unit cube in $L_\mathbb{R} = L \otimes \mathbb{R} = \mathbb{R}^3$, i.e.

$$\square := \left\{ (u_1, u_2, u_3) \in \mathbb{R}^3 \,\middle|\, 0 \leq u_i < 1 \right\}.$$

Since $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$, one can see that $\square$ contains $r - 1$ lattice points

$$v_i = \tfrac{1}{r}(\overline{i\alpha_1}, \overline{i\alpha_2}, \overline{i\alpha_3})$$

for $1 \leq i < r$ where $^{-}$ denotes the residue modulo r. In the case of type $\frac{1}{r}(1, a, r-a)$, these lattice points lie on the plane $y + z = 1$ and they are all the nonzero lattice points in $\square$ except e_1, e_2, e_3.

2.4 Generalized G-graphs

Definition 2.4.1. A *(generalized) G-graph* Γ is a subset of Laurent monomials in $\mathbb{C}[x^{\pm}, y^{\pm}, z^{\pm}]$ satisfying:

(i) $\mathbf{1} \in \Gamma$.

(ii) $\mathrm{wt}\colon \Gamma \to G^{\vee}$ is bijective, i.e. for each weight $\rho \in G^{\vee}$, there exists a unique Laurent monomial $\mathbf{m}_\rho \in \Gamma$ whose weight is ρ.

(iii) if $\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{m}, \mathbf{n} \in \overline{M}_{\geq 0}$, then $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$.

(iv) Γ is *connected* in the sense that for any element $\mathbf{m}_\rho$, there is a (fractional) path from $\mathbf{m}_\rho$ to $\mathbf{1}$ whose steps consist of multiplying or dividing by one of x, y, z in Γ.

For any Laurent monomial $\mathbf{m} \in \overline{M}$, let $\mathrm{wt}_\Gamma(\mathbf{m})$ denote the unique element $\mathbf{m}_\rho$ in Γ whose weight is $\mathrm{wt}(\mathbf{m})$.

Remark 2.4.2. Nakamura G-graphs Γ in [25] are G-graphs in this sense because if a monomial $\mathbf{m} \cdot \mathbf{n}$ is in Γ for two monomials $\mathbf{m}, \mathbf{n} \in \overline{M}_{\geq 0}$, then $\mathbf{m}$ is in Γ. The main difference between Nakamura's definition and ours is that we allow elements to be Laurent monomials, not just genuine monomials. ♦

Example 2.4.3. Let G be the group of type $\frac{1}{7}(1,3,4)$. Then

$$\Gamma_1 = \left\{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\right\},$$
$$\Gamma_2 = \left\{1, z, y, y^2, \tfrac{y^2}{z}, \tfrac{y^3}{z}, \tfrac{y^3}{z^2}\right\}$$

are G-graphs. In Γ_1, $\mathrm{wt}_{\Gamma_1}(x) = \frac{z}{y}$ and $\mathrm{wt}_{\Gamma_1}(y^3) = \frac{z^2}{y^2}$. ♦

As is defined in [25], for a generalized G-graph $\Gamma = \{\mathbf{m}_\rho\}$, define $S(\Gamma)$ to be the subsemigroup of M generated by $\dfrac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}$ for all $\mathbf{m} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. Define a cone $\sigma(\Gamma)$ in $L_{\mathbb{R}} = \mathbb{R}^3$ as follows:

$$\begin{aligned}\sigma(\Gamma) &= S(\Gamma)^{\vee} \\ &= \left\{\mathbf{u} \in L_{\mathbb{R}} \;\middle|\; \left\langle \mathbf{u}, \frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)} \right\rangle \geq 0 \quad \forall \mathbf{m}_\rho \in \Gamma,\ \mathbf{m} \in \overline{M}_{\geq 0}\right\}.\end{aligned}$$

Observe that:

(i) $\sigma(\Gamma) \subset \sigma_+$,

(ii) $\left(\overline{M}_{\geq 0} \cap M\right) \subset S(\Gamma)$,

(iii) $S(\Gamma) \subset \left(\sigma(\Gamma)^\vee \cap M\right)$.

Define two affine toric open sets:

$$\begin{aligned} U(\Gamma) &:= \operatorname{Spec} \mathbb{C}[S(\Gamma)], \\ U^\nu(\Gamma) &:= \operatorname{Spec} \mathbb{C}[\sigma^\vee(\Gamma) \cap M]. \end{aligned}$$

Note that $U^\nu(\Gamma)$ is the normalization of $U(\Gamma)$ and that the torus $\operatorname{Spec} \mathbb{C}[M]$ of $U(\Gamma)$ is isomorphic to $(\mathbb{C}^\times)^3/G$.

Craw, Maclagan and Thomas [5] showed that there exists a torus invariant G-cluster which does not lie over the birational component Y_θ. The following definition is implicit in [5].

Definition 2.4.4. A generalized G-graph Γ is called a *G-iraffe* if the open set $U(\Gamma)$ has a torus fixed point.

Remark 2.4.5. As is standard in toric geometry, note that $U(\Gamma)$ has a torus fixed point if and only if $S(\Gamma) \cap (S(\Gamma))^{-1} = \{\mathbf{1}\}$. The open set $U(\Gamma)$ does not need to have a torus fixed point. In other words, the cone $\sigma(\Gamma)$ is not necessarily a 3-dimensional cone. For counterexamples, see Appendix B. ♦

Example 2.4.6. For the G-graphs in Example 2.4.3,

$$\begin{aligned} \sigma(\Gamma_1) &= \left\{ \mathbf{u} \in \mathbb{R}^3 \,\middle|\, \langle \mathbf{u}, \mathbf{m} \rangle \geq 0, \text{ for all } \mathbf{m} \in \{\tfrac{y^5}{z^2}, \tfrac{z^3}{y^4}, \tfrac{xy}{z}\} \right\}, \\ &= \operatorname{Cone}\Big((1,0,0), \tfrac{1}{7}(3,2,5), \tfrac{1}{7}(1,3,4)\Big), \text{ and} \\ \sigma(\Gamma_2) &= \left\{ \mathbf{u} \in \mathbb{R}^3 \,\middle|\, \langle \mathbf{u}, \mathbf{m} \rangle \geq 0, \text{ for all } \mathbf{m} \in \{\tfrac{y^4}{z^3}, \tfrac{z^4}{y^3}, \tfrac{xz^2}{y^3}\} \right\}, \\ &= \operatorname{Cone}\Big((1,0,0), \tfrac{1}{7}(1,3,4), \tfrac{1}{7}(6,4,3)\Big). \end{aligned}$$

In both cases, they are G-iraffes. One can see that $S(\Gamma_1) = \sigma(\Gamma_1)^\vee \cap M$ and $S(\Gamma_2) = \sigma(\Gamma_2)^\vee \cap M$. ♦

Lemma 2.4.7. *Let Γ be a G-graph. Define*

$$B(\Gamma) := \Big\{ \mathbf{f} \cdot \mathbf{m}_\rho \,\big|\, \mathbf{m}_\rho \in \Gamma, \ \mathbf{f} \in \{x, y, z\} \Big\} \backslash \Gamma.$$

Then the semigroup $S(\Gamma)$ is generated as a semigroup by $\frac{\mathbf{b}}{\mathrm{wt}_\Gamma(\mathbf{b})}$ for all $\mathbf{b} \in B(\Gamma)$. In particular, $S(\Gamma)$ is finitely generated as a semigroup.

Proof. Let S be the subsemigroup of M generated by $\frac{\mathbf{b}}{\mathrm{wt}_\Gamma(\mathbf{b})}$ for all $\mathbf{b} \in B(\Gamma)$ as a semigroup. Clearly, $S \subset S(\Gamma)$. For the inverse inclusion, it is enough to show that the generators of $S(\Gamma)$ are in S.

An arbitrary generator of $S(\Gamma)$ is of the form $\frac{\mathbf{m}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)}$ for some $\mathbf{m} \in \overline{M}_{\geq 0}$, $\mathbf{m}_\rho \in \Gamma$. We may assume that $\mathbf{m} \cdot \mathbf{m}_\rho \notin \Gamma$. In particular, $\mathbf{m} \neq \mathbf{1}$. Since $\mathbf{m}$ has positive degree, there exists $\mathbf{f} \in \{x, y, z\}$ such that $\mathbf{f}$ divides $\mathbf{m}$, i.e. $\frac{\mathbf{m}}{\mathbf{f}} \in \overline{M}_{\geq 0}$ and $\deg(\frac{\mathbf{m}}{\mathbf{f}}) < \deg(\mathbf{m})$. Let $\mathbf{m}_{\rho'}$ denote $\mathrm{wt}_\Gamma(\frac{\mathbf{m}}{\mathbf{f}} \cdot \mathbf{m}_\rho)$. Note that

$$\mathrm{wt}_\Gamma(\mathbf{f} \cdot \mathbf{m}_{\rho'}) = \mathrm{wt}_\Gamma(\mathbf{f} \cdot \frac{\mathbf{m}}{\mathbf{f}} \cdot \mathbf{m}_\rho) = \mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho).$$

Thus

$$\begin{aligned}
\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)} &= \frac{\frac{\mathbf{m}}{\mathbf{f}} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\frac{\mathbf{m}}{\mathbf{f}} \cdot \mathbf{m}_\rho)} \cdot \frac{\mathbf{f} \cdot \mathrm{wt}_\Gamma(\frac{\mathbf{m}}{\mathbf{f}} \cdot \mathbf{m}_\rho)}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)} \\
&= \frac{\frac{\mathbf{m}}{\mathbf{f}} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\frac{\mathbf{m}}{\mathbf{f}} \cdot \mathbf{m}_\rho)} \cdot \frac{\mathbf{f} \cdot \mathbf{m}_{\rho'}}{\mathrm{wt}_\Gamma(\mathbf{f} \cdot \mathbf{m}_{\rho'})}.
\end{aligned}$$

By induction on the degree of monomial $\mathbf{m}$, the assertion is proved. $\square$

2.5 G-graphs and local charts

Let Γ be a G-graph. Define

$$C(\Gamma) := \langle \Gamma \rangle / \langle B(\Gamma) \rangle,$$

then it can be seen that $C(\Gamma)$ is a torus invariant G-constellation. Note that $C(\Gamma)$ can be realised as follows: $C(\Gamma)$ is the $\mathbb{C}$-vector space with a basis Γ whose G-action is induced by the G-action on $\mathbb{C}[x, y, z]$ and whose $\mathbb{C}[x, y, z]$-action is given by

$$\mathbf{m} * \mathbf{m}_\rho = \begin{cases} \mathbf{m} \cdot \mathbf{m}_\rho & \text{if } \mathbf{m} \cdot \mathbf{m}_\rho \in \Gamma, \\ 0 & \text{if } \mathbf{m} \cdot \mathbf{m}_\rho \notin \Gamma, \end{cases}$$

for a monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$.

Any submodule $\mathcal{G}$ of $C(\Gamma)$ is determined by a subset $A \subset \Gamma$, which forms a $\mathbb{C}$-basis of $\mathcal{G}$. We give a combinatorial description of submodules of $C(\Gamma)$.

Lemma 2.5.1. *Let A be a subset of Γ. The following are equivalent.*

(i) The set A forms a $\mathbb{C}$-basis of a submodule of $C(\Gamma)$.

(ii) If $\mathbf{m}_\rho \in A$ and $\mathbf{f} \in \{x, y, z\}$, then $\mathbf{f} \cdot \mathbf{m}_\rho \in \Gamma$ implies $\mathbf{f} \cdot \mathbf{m}_\rho \in A$.

Example 2.5.2. From Example 2.4.3, recall the G-graph

$$\Gamma = \{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\},$$

where G is of type $\frac{1}{7}(1,3,4)$. For the element $y + y^2 + \frac{z}{y}$ in $C(\Gamma)$,

$$y * (y + y^2 + \tfrac{z}{y}) = y^2 + 0 + z = y^2 + z \in C(\Gamma).$$

Let $\mathcal{G}$ be the submodule of $C(\Gamma)$ generated by a basis e_1 of $\mathbb{C}\rho_1$. Then one can see that the set $A = \{z, \frac{z}{y}, \frac{z^2}{y}\}$ satisfies the condition (ii) in the lemma above. Indeed, A is a $\mathbb{C}$-basis of $\mathcal{G}$. ♦

Let p be a point in $U(\Gamma)$. Then there exists the evaluation map

$$\mathrm{ev}_p \colon S(\Gamma) \to (\mathbb{C}, \times),$$

which is a semigroup homomorphism.

To assign a G-constellation $C(\Gamma)_p$ to the point p of $U(\Gamma)$, firstly consider the $\mathbb{C}$-vector space with basis Γ whose G-action is induced by the G-action on $\mathbb{C}[x, y, z]$. Endow it with the following $\mathbb{C}[x, y, z]$-action,

$$\mathbf{m} * \mathbf{m}_\rho := \mathrm{ev}_p \left(\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)} \right) \mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho), \tag{2.5.3}$$

for a monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ and an element $\mathbf{m}_\rho$ in Γ.

Lemma 2.5.4. *With the notation as above, we have the following:*

(i) $C(\Gamma)_p$ is a G-constellation for any $p \in U(\Gamma)$.

(ii) For any p, Γ is a $\mathbb{C}$-basis of $C(\Gamma)_p$.

(iii) $C(\Gamma)_p \ncong C(\Gamma)_q$, if p and q are different points in $U(\Gamma)$.

(iv) Let $Z \subset \mathbf{T} = (\mathbb{C}^\times)^3$ be a free G-orbit and p the corresponding point in the torus $\operatorname{Spec} \mathbb{C}[M]$ *of $U(\Gamma)$. Then $C(\Gamma)_p \cong \mathcal{O}_Z$ as G-constellations.*

(v) If Γ is a G-iraffe and p is the torus fixed point of $U(\Gamma)$, then $C(\Gamma)_p \cong C(\Gamma)$.

Proof. From the definition of $C(\Gamma)_p$, The assertions (i), (ii) and (v) follow immediately. The assertion (iii) follows from the fact [3] that points on the affine toric variety $U(\Gamma)$ are in 1-to-1 correspondence with semigroup homomorphisms from $S(\Gamma)$ to $\mathbb{C}$.

It remains to show (iv). Let $Z \subset \mathbf{T} = (\mathbb{C}^\times)^3$ be a free G-orbit and p the corresponding point in $\operatorname{Spec} \mathbb{C}[M] \subset U(\Gamma)$. Define a G-equivariant $\mathbb{C}[x, y, z]$-module homomorphism

$$\mathbb{C}[x, y, z] \to C(\Gamma)_p, \qquad \text{given by } f \mapsto f * \mathbf{1}.$$

One can check the morphism is surjective and whose kernel is equal to the ideal of Z. This proves (iv). $\square$

This is a family of McKay quiver representations in the following sense of [18].

Definition 2.5.5. *A family of representations of a quiver Q with relations over a scheme B* is a representation of Q with relations in the category of locally free sheaves over B.

Definition 2.5.6. A G-graph is said to be *θ-stable* if the G-constellation $C(\Gamma)$ is θ-stable.

Proposition 2.5.7. *Let Γ be a G-iraffe, that is, $U(\Gamma)$ has a torus fixed point. Let Y_θ be the birational component in $\mathcal{M}_\theta$. For a generic θ, assume that $C(\Gamma)$ is θ-stable. Then $C(\Gamma)_p$ is θ-stable for any $p \in U(\Gamma)$. Thus there exists an open immersion*

$$U(\Gamma) = \operatorname{Spec} \mathbb{C}[S(\Gamma)] \hookrightarrow Y_\theta \subset \mathcal{M}_\theta.$$

Proof. Let us assume that the G-constellation $C(\Gamma)$ is θ-stable. Let p be an arbitrary point in $U(\Gamma)$ and $\mathcal{G}$ a submodule of $C(\Gamma)_p$. By the definition of $C(\Gamma)_p$, it is clear that $\mathcal{G}$ is a submodule of $C(\Gamma)$. Since $C(\Gamma)$ is θ-stable, $\theta(\mathcal{G}) > 0$, and thus $C(\Gamma)_p$ is θ-stable.

Now we introduce deformation theory of the G-constellation in $\mathcal{M}_\theta$. Deforming $C(\Gamma)$ involves $3r$ parameters $\left\{x_\rho, y_\rho, z_\rho \,\middle|\, \rho \in G^\vee\right\}$

$$\begin{cases} x * \mathbf{m}_\rho = x_\rho \operatorname{wt}_\Gamma(x \cdot \mathbf{m}_\rho), \\ y * \mathbf{m}_\rho = y_\rho \operatorname{wt}_\Gamma(y \cdot \mathbf{m}_\rho), \\ z * \mathbf{m}_\rho = z_\rho \operatorname{wt}_\Gamma(z \cdot \mathbf{m}_\rho), \end{cases}$$

such that the following quadrics vanish:

$$\begin{cases} x_\rho y_{\mathrm{wt}(x\cdot\mathbf{m}_\rho)} - y_\rho x_{\mathrm{wt}(y\cdot\mathbf{m}_\rho)}, \\ x_\rho z_{\mathrm{wt}(x\cdot\mathbf{m}_\rho)} - z_\rho x_{\mathrm{wt}(z\cdot\mathbf{m}_\rho)}, \\ y_\rho z_{\mathrm{wt}(y\cdot\mathbf{m}_\rho)} - z_\rho y_{\mathrm{wt}(y\cdot\mathbf{m}_\rho)}. \end{cases} \tag{2.5.8}$$

Since Γ is a $\mathbb{C}$-basis, for $\mathbf{f} \in \{x, y, z\}$, $\mathbf{f}_\rho = 1$ if $\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho) = \mathbf{f}\cdot\mathbf{m}_\rho$. Define a subset of the $3r$ parameters

$$\Lambda(\Gamma) := \left\{\mathbf{f}_\rho \,\middle|\, \mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho) = \mathbf{f}\cdot\mathbf{m}_\rho,\ \mathbf{f}_\rho \in \{x_\rho, y_\rho, z_\rho\}\right\}.$$

Define an affine scheme $D(\Gamma)$ whose coordinate ring is

$$\mathbb{C}\left[x_\rho, y_\rho, z_\rho \,\middle|\, \rho \in G^\vee\right]/I_\Gamma$$

where $I_\Gamma = \left\langle \text{ the quadrics in (2.5.8)}, \mathbf{f} - 1 \,\middle|\, \mathbf{f} \in \Lambda(\Gamma)\right\rangle$.

By King's GIT [18], the affine scheme $D(\Gamma)$ is an open set of $\mathcal{M}_\theta$ which contains the point corresponding to $C(\Gamma)$. More precisely, for a θ-stable G-graph Γ, we have an affine open set $\widetilde{U_\Gamma}$ in the McKay quiver representation space $\operatorname{Rep} G$, which is defined by $\mathbf{f}_\rho$ to be nonzero for all $\mathbf{f}_\rho \in \Lambda(\Gamma)$. Note that $\widetilde{U_\Gamma}$ is $\mathrm{GL}(\delta)$-invariant and that any point in $\widetilde{U_\Gamma}$ is θ-stable. Since the quotient map $\operatorname{Rep}^s G \to \mathcal{M}_\theta$ is a geometric quotient, by GIT (see Remark 2.1.5), it follows that

$$\widetilde{U_\Gamma} /\!\!/ \mathrm{GL}(\delta) = \operatorname{Spec} \mathbb{C}[\widetilde{U_\Gamma}]^{\mathrm{GL}(\delta)}$$

is an open set in $\mathcal{M}_\theta$. On the other hand, after changing basis, we can set $\mathbf{f}_\rho \in \Lambda(\Gamma)$ to be 1 for all $\mathbf{f}_\rho \in \Lambda(\Gamma)$. One can see that this gives a slice[2] so that $D(\Gamma)$ is isomorphic to $\operatorname{Spec} \mathbb{C}[\widetilde{U_\Gamma}]^{\mathrm{GL}(\delta)}$.

Note that there is a $\mathbb{C}$-algebra epimorphism from $\mathbb{C}[D(\Gamma)]$ to $\mathbb{C}[S(\Gamma)]$ defined by

$$\mathbf{f}_\rho \mapsto \frac{\mathbf{f}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{f}\cdot\mathbf{m}_\rho)},$$

for $\mathbf{f}_\rho \in \{x_\rho, y_\rho, z_\rho\}$. It follows that $U(\Gamma)$ is a closed subscheme of $D(\Gamma)$.

As Craw, Maclagan, and Thomas [4] proved that the birational component

[2] First, see that $\mathbb{C}[\widetilde{U_\Gamma}] = \operatorname{Rep} G[\Lambda(\Gamma)^{-1}]$. Note that $\mathrm{GL}(\delta)$-invariants in $\mathbb{C}[\widetilde{U_\Gamma}]$ are generated by cycles with inverting the arrows in $\Lambda(\Gamma)$. Assume that a is the linear map corresponding to an arrow from ρ to ρ'. For ρ, ρ', there exists an undirected path $\mathbf{p}_a$ in $\Lambda(\Gamma) \cap \Lambda(\Gamma)^{-1}$ from ρ to ρ', that is unique up to the commutation relations. This means that $a\mathbf{p}_a^{-1}$ is $\mathrm{GL}(\delta)$-invariants. From this, one can show that there exists an algebra isomorphism between $\mathbb{C}[D(\Gamma)]$ to $\mathbb{C}[\widetilde{U_\Gamma}]^{\mathrm{GL}(\delta)}$ defined by $a \mapsto a\mathbf{p}_a^{-1}$.

Y_θ is a unique irreducible component of $\mathcal{M}_\theta$ containing torus T which is isomorphic to $(\mathbb{C}^\times)^3/G$ as an algebraic group, $Y_\theta \cap D(\Gamma)$ is a unique irreducible component of $D(\Gamma)$ which contains the torus T. Note that $Y_\theta \cap D(\Gamma)$ is reduced by Remark 2.2.14.

We now prove that the morphism $U(\Gamma) \to D(\Gamma) \subset \mathcal{M}_\theta$ induces an isomorphism from the torus $\operatorname{Spec} \mathbb{C}[M]$ onto the torus T of Y_θ. In other words, $U(\Gamma)$ contains the torus T of Y_θ. Let ψ denote the restriction of the morphism to $\operatorname{Spec} \mathbb{C}[M]$. First note that T represents G-constellations whose support is in $\mathbf{T} = (\mathbb{C}^\times)^3$. Let p be a point in the torus $\operatorname{Spec} \mathbb{C}[M] \subset U(\Gamma)$ with the corresponding free G-orbit Z. By Lemma 2.5.4, the G-constellation $C(\Gamma)_p$ over p is isomorphic to $\mathcal{O}_Z$. Thus ψ maps $\operatorname{Spec} \mathbb{C}[M]$ into T. On the other hand, Lemma 2.2.12 shows that any G-constellation whose support is a free G-orbit Z in $\mathbf{T}$ is isomorphic to $\mathcal{O}_Z$. From this, it follows that ψ is a bijective morphism between the two tori. As ψ is a group homomorphism by the construction of $C(\Gamma)_p$, ψ is an isomorphism between $\operatorname{Spec} \mathbb{C}[M]$ and T.

Remember that $U(\Gamma)$ is reduced and irreducible as it is defined by an affine semigroup algebra $\mathbb{C}[S(\Gamma)]$. Note that $U(\Gamma)$ is in the component $Y_\theta \cap D(\Gamma)$ because $U(\Gamma)$ is a closed subset of $D(\Gamma)$ containing T. Since both are of the same dimension, $U(\Gamma)$ is equal to $Y_\theta \cap D(\Gamma)$. Thus there exists an open immersion from $U(\Gamma)$ to Y_θ. $\square$

2.6 G-iraffes and torus fixed points in Y_θ

In this section, we present a 1-to-1 correspondence between the set of torus fixed points in Y_θ and the set of θ-stable G-iraffes.

For a genuine monomial $\mathbf{m} \in \overline{M}_{\geq 0}$, let $\mathbf{m}_{(\rho)}$ denote the path induced by $\mathbf{m}$ in the McKay quiver from the vertex ρ. In other words, $\mathbf{m}_{(\rho)}$ is the linear map induced by the action of the monomial $\mathbf{m}$ on the vector space $\mathbb{C}\rho$.

An *undirected path* in the McKay quiver is a path in the underlying graph of the McKay quiver. For a G-constellation $\mathcal{F}$, an undirected path in the McKay quiver is said to be *defined* if the linear maps corresponding to the opposite-directed arrows in the path are nonzero in $\mathcal{F}$.

Definition 2.6.1. A defined undirected path in the McKay quiver is *of type* $\mathbf{m}$ for a Laurent monomial $\mathbf{m} \in \overline{M}$ where $\mathbf{m}$ is the Laurent monomial obtained by forgetting outgoing vertices.

Example 2.6.2. Let G be the group of type $\frac{1}{7}(1,3,4)$. Consider the G-graph

$$\Gamma = \{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\}.$$

The torus invariant G-constellation $C(\Gamma)$ has the following configurations:

$$\begin{tikzcd}
\rho_2 \arrow[r, "y"] & \rho_5 & & & \\
& \rho_1 \arrow[u, "z"] \arrow[r, "y"] & \rho_4 & & \\
& & \rho_0 \arrow[u, "z"] \arrow[r, "y"] & \rho_3 \arrow[r, "y"] & \rho_6
\end{tikzcd}
\qquad
\begin{tikzcd}
\frac{z^2}{y^2} \arrow[r, "y"] & \frac{z^2}{y} & & & \\
& \frac{z}{y} \arrow[u, "z"] \arrow[r, "y"] & z & & \\
& & 1 \arrow[u, "z"] \arrow[r, "y"] & y \arrow[r, "y"] & y^2
\end{tikzcd}$$

where the marked arrows are nonzero and the others are all zero. The path from 1 to y^2 is induced by y^2 at ρ_0, whose type is y^2. The undirected path from ρ_2 to ρ_4 is a defined undirected path of type $\frac{y^2}{z}$ because the path consists of nonzero linear maps:

$$\rho_2 \xrightarrow{y} \rho_5 \xleftarrow{z} \rho_1 \xrightarrow{y} \rho_4.$$

However, the following undirected path of the same type $\frac{y^2}{z}$ from ρ_2 to ρ_4

$$\rho_2 \xrightarrow{y} \rho_5 \xrightarrow{y} \rho_1 \xleftarrow{z} \rho_4$$

is not defined because the third arrow is zero in $C(\Gamma)$. ♦

Remark 2.6.3. Let $\mathbf{p}$ be a nonzero path induced by a genuine monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ from ρ_i. If $\mathbf{q}$ is a path induced by a genuine monomial $\mathbf{n} \in \overline{M}_{\geq 0}$ from ρ_i with the condition that $\mathbf{n}$ divides $\mathbf{m}$, then the path $\mathbf{q}$ is nonzero. ♦

Lemma 2.6.4. *Let $\mathcal{F}$ be a torus invariant G-constellation. Then there are no defined (undirected) cycles of type $\mathbf{m}$ with $\mathbf{m} \neq \mathbf{1}$.*

Proof. For a contradiction, suppose that there is a defined cycle of type $\mathbf{m} \neq \mathbf{1}$. Then $\mathbf{m}$ is a G-invariant Laurent monomial.

We may assume that the cycle is a cycle around ρ_0 of type $\mathbf{m} = x^{m_1}y^{m_2}z^{m_3}$. A point $(t_1, t_2, t_3) \in \mathbf{T} = (\mathbb{C}^\times)^3$ acts on the cycle by a scalar multiplication of ${t_1}^{m_1}{t_2}^{m_2}{t_3}^{m_3}$. Since $\mathbf{m} \neq \mathbf{1}$, there exists $t \in \mathbf{T}$ such that ${t_1}^{m_1}{t_2}^{m_2}{t_3}^{m_3} \neq 1$, i.e. $t^*(\mathcal{F})$ is not isomorphic to $\mathcal{F}$. Therefore $\mathcal{F}$ is not torus invariant. □

In Section 2.5, we proved that if Γ is a θ-stable G-iraffe, then $C(\Gamma)$ is a torus invariant G-constellation over Y_θ and the corresponding point is fixed by its algebraic torus. Clearly, two different G-iraffes Γ, Γ' give non-isomorphic G-constellations $C(\Gamma)$, $C(\Gamma')$. Moreover, we now prove that for any torus fixed point $p \in Y_\theta$, the corresponding G-constellation is isomorphic to $C(\Gamma)$ for some G-iraffe Γ.

Let p be a torus fixed point in Y_θ. There exists a one parameter subgroup

$$\lambda^u\colon \mathbb{C}^\times \longrightarrow T \subset Y_\theta$$

with $\lim_{t\to 0} \lambda^u(t) = p$. Since Y_θ is the fine moduli space of θ-stable G-constellations, we have a family $\mathcal{U}$ of θ-stable G-constellations over $\mathbb{A}^1_{\mathbb{C}}$ with the following property: for nonzero $s \in \mathbb{A}^1_{\mathbb{C}}$ and the point $q := \lambda^u(s)$, the G-constellation $\mathcal{U}_s$ over s is isomorphic to $\mathcal{O}_Z$ where Z is the free G-orbit in $\mathbf{T}$ corresponding to the point q. In particular, the support of the G-constellation $\mathcal{U}_s$ is in the torus $\mathbf{T} = (\mathbb{C}^\times)^3 \subset \mathbb{C}^3$.

Let $\mathcal{F}$ be the θ-stable G-constellation over $0 \in \mathbb{A}^1$. Let us define a subset Γ of Laurent monomials to be

$$\Gamma = \left\{ \mathbf{m} \in \overline{M} \,\middle|\, \exists \text{ a defined nonzero undirected path in } \mathcal{F} \text{ of type } \mathbf{m} \text{ from } \rho_0 \right\}.$$

Firstly, we prove that Γ is a G-graph. Clearly, Γ contains $\mathbf{1}$. Since θ is generic and $\mathcal{F}$ is θ-stable, there exists a nonzero undirected defined path from ρ_0 to ρ so there is a Laurent monomial $\mathbf{m}_\rho$ in Γ for each $\rho \in G^\vee$. The Laurent monomial $\mathbf{m}_\rho$ is unique: suppose there exists a defined path of type $\mathbf{n}_\rho$ from ρ_0 to ρ, and then there exists a defined cycle of type $\frac{\mathbf{m}_\rho}{\mathbf{n}_\rho}$ at ρ_0, which implies $\mathbf{n}_\rho = \mathbf{m}_\rho$ by Lemma 2.6.4. It remains to show the condition (c) of Definition 2.4.1. We need the following lemma:

Lemma 2.6.5. *With the notation as above, let* $\mathbf{p}$ *and* $\mathbf{q}$ *be two defined (undirected) paths of the same type* $\mathbf{m}$ *from* ρ *to* ρ' *for some Laurent monomial* $\mathbf{m} \in \overline{M}$*. Then, in* $\mathcal{F}$*,*

$$\mathbf{p} * e_\rho = \mathbf{q} * e_\rho$$

where e_ρ *is a basis of* $\mathbb{C}\rho$*.*

Proof. Firstly, note that if $\mathbf{m}$ is a genuine monomial, then the assertion follows from the $\mathbb{C}[x, y, z]$-module structure.

Let $\mathbf{m}$ be a Laurent monomial. There exists a genuine monomial $\mathbf{n} \in \overline{M}_{\geq 0}$ so that $\mathbf{n} \cdot \mathbf{m}$ is a genuine monomial with $\mathbf{n}$ nonzero on $\lambda^u(\mathbb{C}^\times)$. Since two paths $\mathbf{p} * e_\rho$ and $\mathbf{q} * e_\rho$ are of type $\mathbf{m} \cdot \mathbf{n}$, we have

$$\mathbf{n}_{(\rho')} * \mathbf{p} * e_\rho = \mathbf{n}_{(\rho')} * \mathbf{q} * e_\rho. \tag{2.6.6}$$

Since (2.6.6) implies $\mathbf{p} * e_\rho = \mathbf{q} * e_\rho$ in the G-constellation $\mathcal{U}_s$ for nonzero $s \in \mathbb{A}^1$, the assertion is proved by flatness of the family $\mathcal{U}$. □

To show that Γ satisfies the condition (c) of Definition 2.4.1, suppose that $\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$ for $\mathbf{m}_\rho \in \Gamma$ and $\mathbf{m}, \mathbf{n} \in \overline{M}_{\geq 0}$. We need to show that $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$.

By the definition of Γ, there exist nonzero (undirected) paths $\mathbf{p}$ of type $\mathbf{m} \cdot \mathbf{n} \cdot \mathbf{m}_\rho$ and $\mathbf{q}$ of type $\mathbf{m}_\rho$. By Lemma 2.6.5, it follows that the defined undirected path $\mathbf{m}_{(\rho'')} * \mathbf{n}_{(\rho')} * \mathbf{q}$ is nonzero as it is of the same type as bp. This implies that the defined undirected path $\mathbf{n}_{(\rho')} * \mathbf{q}$ is nonzero. Thus $\mathbf{n} \cdot \mathbf{m}_\rho \in \Gamma$.

Proposition 2.6.7. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite cyclic group of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. For a generic parameter θ, there is a 1-to-1 correspondence between the set of torus fixed points in the birational component Y_θ and the set of θ-stable G-iraffes.*

Proof. From the argument above, we have shown that there exists a G-graph Γ for each torus fixed point p. Using Lemma 2.6.5, one can easily show that $C(\Gamma)$ is actually isomorphic to $\mathcal{F}$ as a G-constellation. In particular, $C(\Gamma)$ lies over $p \in Y_\theta$, and hence $U(\Gamma)$ contains the torus fixed point p. Thus Γ is a G-iraffe.

Let Γ be a θ-stable G-iraffe. By Proposition 2.5.7 and Lemma 2.5.4, we can see that $C(\Gamma)$ lies over Y_θ for a G-graph Γ if Γ is a G-iraffe. Thus we have a torus fixed point p representing the isomorphism class of $C(\Gamma)$. $\square$

Corollary 2.6.8. *Let Γ be a G-graph. Then $C(\Gamma)$ lies over the birational component Y_θ if and only if Γ is a G-iraffe.*

Theorem 2.6.9. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be a finite diagonal group and θ a generic GIT parameter for G-constellations. Assume that $\mathfrak{G}$ is the set of all θ-stable G-iraffes.*

(i) The birational component Y_θ of $\mathcal{M}_\theta$ is isomorphic to the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{G}} U(\Gamma)$.

(ii) The normalization of Y_θ is isomorphic to the normal toric variety whose toric fan consists of the full dimensional cones $\sigma(\Gamma)$ for $\Gamma \in \mathfrak{G}$ and their faces.

Proof. Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. Consider the lattice

$$L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3).$$

Let Y_θ be the birational component of the moduli space of θ-stable G-constellations and Y_θ^ν the normalization of Y_θ. Let Y denote the not-necessarily-normal toric variety $\bigcup_{\Gamma \in \mathfrak{G}} U(\Gamma)$. Define the fan Σ in $L_\mathbb{R}$ whose full dimensional cones are $\sigma(\Gamma)$ for $\Gamma \in \mathfrak{G}$. One can see that the corresponding toric variety $Y^\nu := X_\Sigma$ is the normalization of Y.

Since Y_θ^ν is a normal toric variety, it is covered by toric affine open sets U_i with the torus fixed point p_i in U_i. Let q_i be the image of p_i under the normalization. As each q_i is a torus fixed point, it follows from Proposition 2.6.7 that there is a (unique) G-iraffe $\Gamma_i \in \mathfrak{G}$ with $C(\Gamma_i)$ isomorphic to the G-constellation represented by q_i.

By Proposition 2.5.7, for each G-iraffe $\Gamma \in \mathfrak{G}$, there is an open immersion of $U(\Gamma)$ into Y_θ. Thus we have an open immersion $\psi\colon Y \to Y_\theta$ and the image $\psi(Y)$ contains all torus fixed points of Y_θ.

The induced morphism $\psi^\nu\colon Y^\nu \to Y_\theta^\nu$ is an open embedding. Note that the numbers of full dimensional cones are the same. Thus ψ^ν should be an isomorphism. This proves (ii).

To show (i), suppose that $Y_\theta \setminus \psi(Y)$ is nonempty so it contains a torus orbit O of dimension $d \geq 1$. Since the normalization morphism is torus equivariant and surjective, there exists a torus orbit O' in $Y_\theta^\nu = Y^\nu$ of dimension d which is mapped to the torus orbit O. At the same time, from the fact that Y^ν is the normalization of Y and that the normalization morphism is finite, it follows that the image of O' is a torus orbit of dimension d, so the image is O. Thus O is in $\psi(Y)$, which is a contradiction. □

Corollary 2.6.10. *With notation as Theorem 2.6.9, Y_θ is a normal toric variety if and only if $S(\Gamma) = \sigma(\Gamma)^\vee \cap M$ for all $\Gamma \in \mathfrak{G}$.*

Remark 2.6.11 (Link to [4])**.** In general, it is hard to find the set of all θ-stable G-iraffes in practice.

Craw, Maclagan, and Thomas [4] described Y_θ using a certain polyhedron P_θ. The vertices $\mathbf{v}_\alpha$ of the polyhedron P_θ correspond to fixed points p_α of the torus action. For each vertex $\mathbf{v}_\alpha$, they define a semigroup A_α such that $\operatorname{Spec} \mathbb{C}[A_\alpha]$ gives an affine open set through p_α.

In our description, since each torus fixed point p_α represents the isomorphism class of a θ-stable torus invariant G-constellation lying over Y_θ, we have a unique G-iraffe Γ_α and the semigroup $S(\Gamma_\alpha)$. We expect that our semigroup $S(\Gamma_\alpha)$ is equal to the semigroup A_α. ♦

Chapter 3

Weighted blowups and economic resolutions

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r, i.e. G is the subgroup generated by the diagonal matrix $\mathrm{diag}(\epsilon, \epsilon^a, \epsilon^{r-a})$ where ϵ is a primitive rth root of unity. The quotient variety $X = \mathbb{C}^3/G$ has terminal singularities and has no crepant resolution. However, there exist a special kind of toric resolutions, which can be obtained by a sequence of weighted blowups. In this section, we review the notion of toric weighted blowups and define round down functions which are used for finding admissible G-iraffes.

3.1 Background: Birational geometry

In this section, we collect various facts from birational geometry. Most of these are taken from [27] and [28].

Definition 3.1.1. Let X be a normal quasiprojective variety.

(i) A Weil divisor D on X is said to be $\mathbb{Q}$-*Cartier* if the Weil divisor rD is Cartier for some integer $r \geq 1$.

(ii) The variety X is said to be $\mathbb{Q}$-*factorial* if every Weil divisor on X is $\mathbb{Q}$-Cartier.

Definition 3.1.2. Let X be a normal quasiprojective variety. We say that X has *terminal singularities* (resp. *canonical singularities*) if it satisfies the following conditions:

(i) the canonical divisor K_X is $\mathbb{Q}$-Cartier.

(ii) if $\varphi\colon Y \to X$ is a resolution with E_i prime exceptional divisors such that

$$K_Y \sim_{\mathbb{Q}} \varphi^*(K_X) + \sum a_i E_i,$$

then $a_i > 0$ (resp. ≥ 0) for all i.

In the definition above, a_i is called the *discrepancy* of E_i. A *crepant resolution* φ of X is a resolution with all discrepancies zero. In particular, X is canonical.

Remark 3.1.3. If a variety X has terminal singularities, then its singular locus has codimension ≥ 3. In particular, terminal singularities in dimension 2 are smooth and terminal singularities in dimension 3 are isolated. ⧫

Remark 3.1.4. For a smooth variety X, let $\varphi\colon Y \to X$ be a projective birational morphism with Y normal. Then the discrepancy of every prime exceptional divisor is ≥ 1. ⧫

Example 3.1.5. Let X be a smooth surface. Suppose that $\varphi\colon Y \to X$ is the blow up of a point in X with exceptional divisor $E \cong \mathbb{P}^1$. It is easy to check that the self intersection number of E is $E^2 = -1$. Assume that the discrepancy of E is a, i.e

$$K_Y = \varphi^*(K_X) + aE.$$

By adjunction, we get

$$-2 = \deg(K_{\mathbb{P}^1}) = (K_Y + E) \cdot E = (a+1)E^2 = -a - 1.$$

It follows that $a = 1$. ⧫

Remark 3.1.6. In the surface case, it is well known that a canonical singularity is analytically isomorphic to a quotient singularity $\mathbb{C}^2/G$ with a finite group $G \subset \mathrm{SL}_2(\mathbb{C})$.

Let G be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ and X the quotient variety $\mathbb{C}^2/G$. Suppose that $\varphi\colon Y \to X$ is the minimal resolution of X. The following are known:

(i) the exceptional locus $\mathrm{Exc}(\varphi)$ of φ is a tree of (-2)-curves.

(ii) the dual graph of the exceptional curves is a Dynkin diagram of ADE type.

The *type* of the group $G \subset \mathrm{SL}_2(\mathbb{C})$ is the type of the Dynkin diagram in (ii). ⧫

Example 3.1.7. Consider the finite subgroup G in $\mathrm{GL}_2(\mathbb{C})$ of type $\frac{1}{r}(1,1)$. The invariant ring in $\mathbb{C}[x,y]$ is

$$\mathbb{C}[x,y]^G = \mathbb{C}\left[x^r, x^{r-1}y, \ldots, xy^{r-1}, y^r\right]$$

which is the coordinate ring of the quotient variety $X = \mathbb{C}^2/G$.

One can show that X is isomorphic to the affine cone over the rational normal curve of degree r. The surface X has a resolution $\varphi\colon Y \to X$ with exceptional divisor $E \cong \mathbb{P}^1$ satisfying $\mathcal{O}_E(-E) \cong \mathcal{O}_{\mathbb{P}^1}(r)$. By the adjunction formula, we have $\mathcal{O}_E(K_Y + E) = K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$, and hence

$$K_Y = \varphi^*(K_X) - \tfrac{r-2}{r}E.$$

Thus the quotient X is not canonical if $r \geq 3$. ◆

The following proposition is well known.

Proposition 3.1.8. *Let X be a $\mathbb{Q}$-factorial variety. Suppose $\varphi\colon Y \to X$ is a resolution of X.*

(i) The exceptional locus of φ has pure codimension 1, i.e. $\mathrm{Exc}(\varphi)$ *is a divisor.*

(ii) If X has only terminal singularities, then X does not admit nontrivial crepant resolutions.

Birational geometry of toric varieties

Let L be a lattice of rank n and M the dual lattice of L. As in Section 2.3, M can be considered as the monomial lattice.

Let σ be a cone in $L \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a primitive element $v \in L \cap \sigma$. The *barycentric subdivision* $\sigma[v]$ of σ at v is the minimal fan containing all cones $\mathrm{Cone}(\tau, v)$ where τ varies all subcone of σ with $v \notin \tau$.

The barycentric subdivision induces a toric morphism

$$X_{\sigma[v]} \longrightarrow U_\sigma.$$

The following proposition is well known in toric geometry (see e.g. [3]).

Proposition 3.1.9. *Let $\Sigma := \sigma[v]$ be the barycentric subdivision of a cone σ at v.*

(i) The barycentric subdivision induces a projective toric morphism

$$X_\Sigma \longrightarrow U_\sigma.$$

(ii) The set of 1-dimensional cones of Σ *consists of the 1-dimensional cones of* σ *and* $\operatorname{Cone}(v)$*.*

(iii) The torus invariant prime divisor D_v *corresponding to the 1-dimensional cone* $\operatorname{Cone}(v)$ *is a* $\mathbb{Q}$*-Cartier divisor on* X_Σ*.*

Example 3.1.10. Let L be the standard lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ with the standard basis e_1, e_2, e_3. Consider the cone $\sigma = \operatorname{Cone}(e_1, e_2, e_2+e_3, e_1+e_3)$. Set $v_1 := e_1$, $v_2 := e_2$, $v_3 := e_1 + e_2 + e_3$ and let Σ_i denote the barycentric subdivision of σ at v_i.

Note that the variety corresponding to σ is the quadric cone $xz - yt = 0$ in $\mathbb{C}^4$, which is singular at the origin. It is easy to see that the varieties X_{Σ_i} are smooth, so the birational morphisms $X_{\Sigma_i} \longrightarrow U_\sigma$ are resolutions of X.

The birational morphism induced by the subdivision at v_3 is the blow up of the origin with exceptional divisor $E = \mathbb{P}^1 \times \mathbb{P}^1$. However, the birational morphism induced by the subdivision at v_1 does not introduce a new divisor, i.e. the exceptional locus is of codimension ≥ 2. More precisely, the exceptional locus is $\mathbb{P}^1$. One can see that there exists a morphism $X_{\Sigma_3} \to X_{\Sigma_1}$ which induces a projection of E onto one factor of $\mathbb{P}^1 \times \mathbb{P}^1$.

$$
\begin{tikzcd}
& X_{\Sigma_3} \arrow[dl] \arrow[dd] \arrow[dr] & \\
X_{\Sigma_1} \arrow[rr, dotted] \arrow[dr] & & X_{\Sigma_2} \arrow[dl] \\
& U_\sigma &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl] \arrow[dd] \arrow[dr] & \\
\mathbb{P}^1 \arrow[rr, dotted] \arrow[dr] & & \mathbb{P}^1 \arrow[dl] \\
& \{\mathrm{pt}\} &
\end{tikzcd}
$$

Figure 3.1.1: Atiyah flop

Note that the birational map from X_{Σ_1} to X_{Σ_2} is an isomorphism outside of codimension 2. This is the simplest example of a *flop*, which was introduced by Atiyah. ♦

Proposition 3.1.11 (Reid [28])**.** *Let* $X := U_\sigma$ *be the affine toric variety corresponding to a* n*-dimensional cone* σ*. Assume that* K_X *is* $\mathbb{Q}$*-Cartier. Let* Y *be the corresponding toric variety of the barycentric subdivision of* σ *at* v *and* $\varphi\colon Y \longrightarrow X$ *the induced toric morphism. Suppose* v *is an interior lattice point in* σ*. Then*

$$K_Y = \varphi^*(K_X) + (\langle x_1 x_2 \cdots x_n, v\rangle - 1) D_v,$$

i.e. the discrepancy of the exceptional divisor D_v *is* $\langle x_1 x_2 \dots x_n, v\rangle - 1$*.*

Proof. Let $\sigma = \mathrm{Cone}(v_1, v_2, \dots, v_k)$ be a cone in L with v_i primitive vectors and D_i the torus invariant prime divisor corresponding to v_i. Consider the holomorphic n-form on the torus

$$s = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \cdots \wedge \frac{dx_n}{x_n},$$

which can be extended to a rational n-form on X so that it has simple poles along all torus invariant prime divisors on X. Thus

$$K_X + \sum D_i \sim_{\mathbb{Q}} 0.$$

In particular,

$$K_Y + \sum \varphi^{-1}(D_i) + D_v \sim_{\mathbb{Q}} \varphi^* \left(K_X + \sum D_i \right).$$

As $\varphi^*(s)$ has a pole of order $\langle x_1 x_2 \cdots x_n, v \rangle$ along the new divisor D_v,

$$\varphi^* \left(K_X + \sum D_i \right) \sim_{\mathbb{Q}} \sum \varphi^{-1}(D_i) + \langle x_1 x_2 \cdots x_n, v \rangle D_v,$$

which proves the assertion. □

Example 3.1.12. Define the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, r-a)$ with a coprime to r and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ the dual lattice. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$ and σ_+ the cone in $L_{\mathbb{R}}$ generated by e_1, e_2, e_3. Set $v_i := \frac{1}{r}(i, \overline{ai}, r - \overline{ai}) \in L$ for each $1 \leq i \leq r-1$.

Let E_i be the torus invariant prime divisor corresponding to v_i. It can be calculated from Proposition 3.1.11 that the discrepancy of E_i is $\frac{i}{r}$. Note that the subdivision at v_1 gives the smallest discrepancy $\frac{1}{r}$ and that any discrepancy of E_i is less than 1. ♦

Theorem 3.1.13 (Reid [27]). *Let X be the toric variety corresponding to a fan Σ with a lattice L and the dual lattice M. Then X has only terminal singularities (resp. canonical singularities) if and only if any cone $\sigma \in \Sigma$ satisfies the conditions (i) and (ii) (resp. (i) and (iii)):*

(i) there exists an element $\mathbf{m} \in M_{\mathbb{Q}}$ such that $\langle \mathbf{m}, u \rangle = 1$ for any primitive vector u of σ.

(ii) there are no other lattice points in the set $\left\{ u \in \sigma \,\middle|\, \langle \mathbf{m}, u \rangle \leq 1 \right\}$ except vertices.

(iii) there are no other lattice points in the set $\left\{ u \in \sigma \,\middle|\, \langle \mathbf{m}, u \rangle < 1 \right\}$ except the origin.

Remark 3.1.14. Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$ and $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(\alpha_1, \alpha_2, \alpha_3)$. From Theorem 3.1.13, it follows that the quotient singularity $\mathbb{C}^3/G$ has only terminal singularities if and only if there are no nonzero lattice points of L lie on and below the plane $x + y + z = 1$ other than e_1, e_2, e_3. In a similar way, one can see that the quotient singularity $\mathbb{C}^3/G$ has only canonical singularities if and only if there are no nonzero lattice points of L lie below the plane $x + y + z = 1$. ♦

Example 3.1.15. Define the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, r - a)$ with a coprime to r and $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ the dual lattice. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$ and σ_+ the cone in $L_{\mathbb{R}}$ generated by e_1, e_2, e_3.

We now show that the toric variety $X := U_{\sigma_+}$ has only terminal singularities. Consider $\mathbf{m} = xyz \in M_{\mathbb{Q}}$. Note that $\mathbf{m}$ satisfies the condition (i) in Theorem 3.1.13. One can show that

$$\left\{u \in \sigma \,\middle|\, \langle \mathbf{m}, u\rangle \leq 1\right\} = \{0, e_1, e_2, e_3\}$$

so it follows that X has only terminal singularities.

In addition, since all quotient singularities are $\mathbb{Q}$-factorial, X does not admit crepant resolutions by Proposition 3.1.8. ♦

In the example above, we have seen that the quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if G is the group of type $\frac{1}{r}(1, a, r - a)$ with a coprime to r. Moreover the following theorem says that there is essentially only one case.

Theorem 3.1.16 (Morrison and Stevens [23])**.** *A 3-fold cyclic quotient singularity $X = \mathbb{C}^3/G$ has terminal singularities if and only if $G \subset \mathrm{GL}_3(\mathbb{C})$ is the subgroup of type $\frac{1}{r}(1, a, r - a)$ with a coprime to r.*

3.2 Weighted blowups and round down functions

Define the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, a, r - a)$ and set $\overline{L} = \mathbb{Z}^3 \subset L$. Consider two dual lattices $M = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ and $\overline{M} = \mathrm{Hom}_{\mathbb{Z}}(\overline{L}, \mathbb{Z})$. Note that a (Laurent) monomial $\mathbf{m} \in \overline{M}$ is invariant under G if and only if $\mathbf{m}$ is in M. Let $\{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{Z}^3$ and σ_+ the cone in $L_{\mathbb{R}}$ generated by e_1, e_2, e_3. Then $\operatorname{Spec} \mathbb{C}[\sigma_+^\vee \cap M]$ is the quotient variety $X = \mathbb{C}^3/G$. Set $v = \frac{1}{r}(1, a, r - a) \in L$, which corresponds to the exceptional divisor of the smallest discrepancy (see Example 3.1.12). Define three cones

$$\sigma_1 = \mathrm{Cone}(v, e_2, e_3), \quad \sigma_2 = \mathrm{Cone}(e_1, v, e_3), \quad \sigma_3 = \mathrm{Cone}(e_1, e_2, v),$$

and define Σ to be the fan consisting of the three cones $\sigma_1, \sigma_2, \sigma_3$ and their faces. The fan Σ is the barycentric subdivision of σ_+ at v. Let Y_1 be the toric variety corresponding to the fan Σ together with the lattice L. Define $\varphi\colon Y_1 \to X$ to be the induced toric morphism, which is called *the weighted blowup* of X with weight $(1, a, r-a)$.

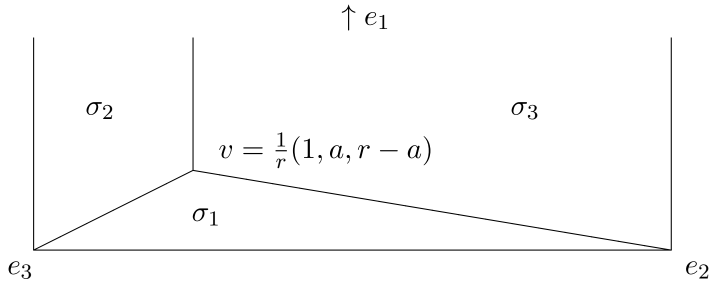


Figure 3.2.1: Weighted blowup of weight $(1, a, r-a)$

Let us consider the sublattice L_2 of L generated by e_1, v, e_3 and let us define $M_2 := \mathrm{Hom}_{\mathbb{Z}}(L_2, \mathbb{Z})$ with dual basis

$$\xi := xy^{-\frac{1}{a}}, \quad \eta := y^{\frac{r}{a}}, \quad \zeta := y^{\frac{a-r}{a}} z.$$

The lattice inclusion $L_2 \hookrightarrow L$ induces a toric morphism

$$\varphi\colon \operatorname{Spec} \mathbb{C}[\sigma_2^\vee \cap M_2] \to U_2 := \operatorname{Spec} \mathbb{C}[\sigma_2^\vee \cap M].$$

Since $\mathbb{C}[\sigma_2^\vee \cap M_2] \cong \mathbb{C}[\xi, \eta, \zeta]$ and the group $G_2 := L/L_2$ is of type $\frac{1}{a}(1, \overline{-r}, \overline{r-a})$ with eigencoordinates ξ, η, ζ, the open subset U_2 has a quotient singularity of type $\frac{1}{a}(1, \overline{-r}, \overline{r-a})$. Note that for $x^{m_1} y^{m_2} z^{m_3} \in \overline{M}_{\geq 0}$,

$$\varphi^*(x^{m_1} y^{m_2} z^{m_3}) = \xi^{m_1} \eta^{\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3} \zeta^{m_3}.$$

Similarly, consider the sublattice L_3 of L generated by e_1, e_2, v. Define the lattice $M_3 := \mathrm{Hom}_{\mathbb{Z}}(L_3, \mathbb{Z})$ with basis

$$\xi_3 := xz^{-\frac{1}{r-a}}, \quad \eta_3 := yz^{\frac{-a}{r-a}}, \quad \zeta_3 := z^{\frac{r}{r-a}}.$$

The open set $U_3 = \operatorname{Spec} \mathbb{C}[\xi_3, \eta_3, \zeta_3]$ has a singularity of type $\frac{1}{r-a}(1, \overline{a}, \overline{r-2a})$ with eigencoordinates ξ_3, η_3, ζ_3 with $G_2 := L/L_3$.

Lastly, consider the sublattice L_1 of L generated by v, e_2, e_3. Let us define

$M_1 := \mathrm{Hom}_{\mathbb{Z}}(L_1, \mathbb{Z})$ with dual basis

$$\xi_1 := xz^{-\frac{1}{r-a}}, \quad \eta_1 := yz^{\frac{-a}{r-a}}, \quad \zeta_1 := z^{\frac{r}{r-a}}.$$

Since $\{v, e_2, e_3\}$ forms a $\mathbb{Z}$-basis of L, i.e. $G_1 = L/L_1$ is the trivial group, the open set $U_1 = \mathrm{Spec}\,\mathbb{C}[\xi_1, \eta_1, \zeta_1]$ is smooth.

Example 3.2.1. Let G be the group of type $\frac{1}{7}(1,3,4)$ as in Example 2.4.3. The fan of the weighted blowup of weight $(1,3,4)$ is shown in Figure 3.2.2.

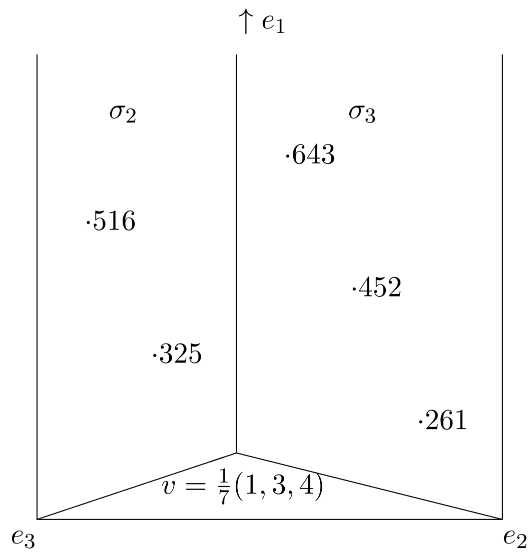


Figure 3.2.2: Weighted blowup of weight $(1,3,4)$

Let U_2 be the affine toric variety corresponding to the cone σ_2 on the left side of $v = \frac{1}{7}(1,3,4)$. Note that U_2 has a quotient singularity of type $\frac{1}{3}(1,2,1)$ with eigencoordinates $xy^{-\frac{1}{3}}, y^{\frac{7}{3}}, y^{-\frac{4}{3}}z$.

Let U_3 be the affine toric variety corresponding to the cone σ_3 on the left side of $v = \frac{1}{7}(1,3,4)$. Note that U_3 has a quotient singularity of type $\frac{1}{3}(1,2,1)$ with eigencoordinates $xz^{-\frac{1}{4}}, yz^{-\frac{3}{4}}, z^{\frac{7}{4}}$.

On the other hand, e_2, e_3, v form a $\mathbb{Z}$-basis of L, so that the affine toric variety corresponding to the cone generated by v, e_2, e_3 is smooth. ♦

Definition 3.2.2 (Round down functions)**.** With the notation above, the *left round down function* $\phi_2 \colon \overline{M} \to M_2$ of the weighted blowup with weight $(1, a, r-a)$ is defined by

$$\phi_2(x^{m_1}y^{m_2}z^{m_3}) = \xi^{m_1}\eta^{\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \rfloor}\zeta^{m_3}.$$

where $\lfloor\ \rfloor$ is round down. In a similar manner, the *right round down function* $\phi_3\colon \overline{M} \to M_3$ of the weighted blowup with weight $(1, a, r-a)$ is defined by

$$\phi_3(x^{m_1}y^{m_2}z^{m_3}) = \xi_3^{m_1}\eta_3^{m_2}\zeta_3^{\lfloor \frac{1}{r}m_1+\frac{a}{r}m_2+\frac{r-a}{r}m_3\rfloor},$$

and the *central round down function* $\phi_1\colon \overline{M} \to M_1$ of the weighted blowup with weight $(1, a, r-a)$ by

$$\phi_1(x^{m_1}y^{m_2}z^{m_3}) = \xi_1^{\lfloor \frac{1}{r}m_1+\frac{a}{r}m_2+\frac{r-a}{r}m_3\rfloor}\eta_1^{m_2}\zeta_1^{m_3}.$$

Remark 3.2.3. Let ϕ_k be a round down function of the weighted blowup with weight $(1, a, r-a)$ as above for $k = 1, 2, 3$. For $\mathbf{m} \in \overline{M}$ and $\mathbf{n} \in M$, we have

$$\phi_k(\mathbf{m}\cdot\mathbf{n}) = \phi_k(\mathbf{m})\cdot\mathbf{n},$$

because M_k contains M as the lattice of G_k invariant monomials, especially, $\mathbf{n}$ is in M_k. Thus the weight of $\phi_k(\mathbf{m}\cdot\mathbf{n})$ and the weight of $\phi_k(\mathbf{m})$ are the same in terms of the G_k action. ♦

Remark 3.2.4. Davis, Logvinenko, and Reid [8] introduce a related construction in a more general setting. ♦

Lemma 3.2.5. *Let ϕ_k be a round down function of the weighted blowup with weight $(1, a, r-a)$ as above for $k = 1, 2, 3$. Let $\mathbf{m} \in \overline{M}$ be a Laurent monomial of weight j. Then we have the following:*

(i) $\phi_2(y\cdot\mathbf{m}) = \phi_2(\mathbf{m})$, *when* $0 \le j < r-a$.

(ii) $\phi_3(z\cdot\mathbf{m}) = \phi_3(\mathbf{m})$, *when* $0 \le j < a$.

(iii) $\phi_1(x\cdot\mathbf{m}) = \phi_1(\mathbf{m})$, *when* $0 \le j < r-1$.

Proof. Let $\mathbf{m} = x^{m_1}y^{m_2}z^{m_3}$ be a Laurent monomial of weight j. To prove (i), assume that $0 \le j < r-a$. This means that

$$0 \le \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3\rfloor < \frac{r-a}{r}.$$

Thus $\phi_2(y\cdot\mathbf{m}) = \phi_2(x^{m_1}y^{m_2+1}z^{m_3}) = \phi_2(x^{m_1}y^{m_2}z^{m_3})$.

The assertions (ii) and (iii) can be proved similarly. □

3.3 Economic resolutions

For each $0 \leq i \leq r$, let $v_i := \frac{1}{r}(i, \overline{ai}, \overline{r - ai})$ be a lattice point in L. The quotient variety $X = \mathbb{C}^3/G$ has a certain toric resolution which was introduced by Danilov [7] (see [28]).

Definition 3.3.1. For the group $G \subset \mathrm{GL}_3(\mathbb{C})$ of type $\frac{1}{r}(1, a, r - a)$, the *economic resolution* of $\mathbb{C}^3/G$ is the toric variety obtained by the consecutive weighted blowups $v_1, v_2, \ldots, v_{r-1}$ from the quotient variety $X = \mathbb{C}^3/G$.

Let $\varphi\colon Y \to X = \mathbb{C}^3/G$ be the economic resolution. For each $1 \leq i < r$, let E_i denote the exceptional divisor of φ corresponding to the lattice point v_i. From toric geometry, we have the following proposition (see Example 3.1.12).

Proposition 3.3.2. *With the notation as above, the economic resolution Y has the following properties:*

(i) Y is smooth and projective over X.

(ii) $K_Y = \varphi^*(K_X) + \sum_{1 \leq i < r} \frac{i}{r} E_i$. In particular, each discrepancy is $0 < \frac{i}{r} < 1$.

Remark 3.3.3. From the fan of Y, we can see that Y can be covered by three open sets U_2, U_3 and U_1, which are the unions of the affine toric varieties corresponding to the cones on the left side of, the right side of, and below the vector $v = \frac{1}{r}(1, a, r - a)$, respectively. Note that U_2 and U_3 are isomorphic to the economic resolutions for the singularity of $\frac{1}{a}(1, \overline{-r}, \overline{r - a})$, of $\frac{1}{r-a}(1, \overline{a}, \overline{-r})$, respectively. ♦

Example 3.3.4. Let G be the group of type $\frac{1}{7}(1, 3, 4)$ as in Example 2.4.3. The fan of the economic resolution of the quotient variety is shown in Figure 3.3.1.

Let U_2 be the toric variety corresponding to the fan consisting of the cones on the left side of $v = \frac{1}{7}(1, 3, 4)$. Note that U_2 is the economic resolution of the quotient $\frac{1}{3}(1, 2, 1)$ which is G_2-Hilb $\mathbb{C}^3$, where G_2 is of type $\frac{1}{3}(1, 2, 1)$.

Let U_3 be the toric variety corresponding to the fan consisting of the cones on the right side of $v = \frac{1}{7}(1, 3, 4)$. Note that U_3 is the economic resolution of the quotient $\frac{1}{4}(1, 3, 1)$ which is G_3-Hilb $\mathbb{C}^3$, where G_3 is of type $\frac{1}{4}(1, 3, 1)$. ♦

3.4 Elephants for the economic resolution

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(1, a, r - a)$. Consider the quotient variety $X = \mathbb{C}^3/G$.

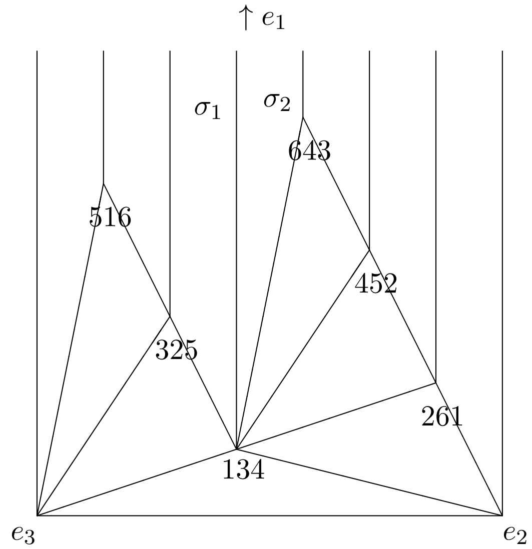


Figure 3.3.1: Fan of the economic resolution for $\frac{1}{7}(1,3,4)$

Let D be the hyperplane section of X defined by $x = 0$, i.e. the Weil divisor defined by $x = 0$. One can see that

$$K_X + D \sim_{\mathbb{Q}} 0$$

from the proof of Proposition 3.1.11. Thus D is an element[3] of the anticanonical system $|-K_X|$. Moreover, D is isomorphic to the quotient $\mathbb{C}^2$ by the group of type $\frac{1}{r}(a, -a)$ so D has an A_{r-1} singularity.

Consider the economic resolution $\varphi\colon Y \to X = \mathbb{C}^3/G$. Let S be the strict transform of D. Then one can show that S is an element of the anticanonical system $|-K_Y|$ and that we have the following diagram:

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ D \arrow[r, hook] & X, \end{tikzcd}$$

where the vertical morphism $S \to D$ is the minimal resolution of D.

It is well known [1, 19] that the minimal resolution of A_{r-1} singularities is isomorphic to the moduli space of θ-stable A-constellations for a generic parameter θ where $A \subset \mathrm{SL}_2(\mathbb{C})$ is the group of type $\frac{1}{r}(1, -1)$. Moreover, the chamber structure of the GIT stability parameter space for A-constellations coincides with the Weyl

[3] Elements of the anticanonical system of a variety X are called *elephants* of X.

chamber structure of type A_{r-1} (see Section 5.1). We expect that the morphism $Y \to X$ might have a modular description as moduli spaces of G-constellations (see Section 5.2).

Chapter 4

Moduli interpretations of economic resolutions

This chapter contains our main theorem. Section 4.1 explains how to find an admissible set $\mathfrak{G}$ of G-iraffes. To find G-iraffes, we use the round down functions introduced in Section 3.2. Section 4.4 describes the universal families over the birational component Y_θ using G-iraffes. In Section 4.2, we show that there exists a stability parameter θ such that G-iraffes in $\mathfrak{G}$ are θ-stable.

4.1 How to find admissible G-iraffes

4.1.1 G-iraffes for $\frac{1}{r}(1, r-1, 1)$

Let G be the finite subgroup in $\mathrm{GL}_3(\mathbb{C})$ of $\frac{1}{r}(1, r-1, 1)$ type, i.e. $a = 1$ or $r-1$. Kędzierski [15] proved that for $G \subset \mathrm{GL}_3(\mathbb{C})$ of type $\frac{1}{r}(1, r-1, 1)$, $G\text{-Hilb}\,\mathbb{C}^3$ is isomorphic to the economic resolution of the quotient variety $\mathbb{C}^3/G$.

Theorem 4.1.1 (Kędzierski [15]). *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with $a = 1$ or $r-1$. Then $G\text{-Hilb}\,\mathbb{C}^3$ is isomorphic to the economic resolution of the quotient variety $\mathbb{C}^3/G$. In particular, $G\text{-Hilb}\,\mathbb{C}^3$ is nonsingular and irreducible.*

For each $0 \le i \le r$, set $v_i = \frac{1}{r}(i, r-i, i)$. The fan corresponding to $G\text{-Hilb}\,\mathbb{C}^3$ consists of the following $2r-1$ maximal cones and their faces:

$$\begin{aligned} \sigma_i &= \mathrm{Cone}(e_1, v_{i-1}, v_i) &\qquad \text{for } 1 \le i \le r, \\ \sigma_{r+i} &= \mathrm{Cone}(e_3, v_{i-1}, v_i) &\qquad \text{for } 1 \le i \le r-1. \end{aligned}$$

Each maximal cone has a corresponding (Nakamura) G-graph:

$$\Gamma_i = \{1, y, y^2, \dots, y^{i-1}, z, z^2, \dots, z^{r-i}\} \quad \text{for } 1 \le i \le r,$$
$$\Gamma_{r+i} = \{1, y, y^2, \dots, y^{i-1}, x, x^2, \dots, x^{r-i}\} \quad \text{for } 1 \le i \le r-1,$$

with $S(\Gamma_j) = \sigma_j^\vee \cap M$ for $1 \le j \le 2r-1$. From the fact that each cone σ_j is 3-dimensional, it is immediate that these G-graphs are G-iraffes.

Example 4.1.2. Let G be the finite group of type $\frac{1}{2}(1,1,1)$. Set $v = \frac{1}{2}(1,1,1)$. Note that the economic resolution Y of $X = \mathbb{C}^3/G$ is the weighted blowup of X with weight $(1,1,1)$. Then the maximal cones of Y are

$$\sigma_1 = \mathrm{Cone}(v, e_2, e_3), \quad \sigma_2 = \mathrm{Cone}(e_1, v, e_3), \quad \sigma_3 = \mathrm{Cone}(e_1, e_2, v),$$

and the corresponding G-iraffes Γ_i to σ_i are

$$\Gamma_1 = \{1, x, x^2\}, \qquad \Gamma_2 = \{1, y, y^2\}, \qquad \Gamma_3 = \{1, z, z^2\}.$$

Let us consider the left round down function ϕ_2, the right round down function ϕ_3 and the central round down function ϕ_1 corresponding to the weighted blowup with weight $(1,1,1)$. Then

$$\Gamma_1 = \left\{\mathbf{m} \in \overline{M} \,\middle|\, \phi_1(\mathbf{m}) = \mathbf{1}\right\},$$
$$\Gamma_2 = \left\{\mathbf{m} \in \overline{M} \,\middle|\, \phi_2(\mathbf{m}) = \mathbf{1}\right\},$$
$$\Gamma_3 = \left\{\mathbf{m} \in \overline{M} \,\middle|\, \phi_3(\mathbf{m}) = \mathbf{1}\right\}.$$

♦

Example 4.1.3. Let G be the finite group of type $\frac{1}{3}(1,2,1)$. Set $v_1 = \frac{1}{3}(1,2,1)$ and $v_2 = \frac{1}{3}(2,1,2)$. In this example, let ξ, η, ζ be the coordinates of $\mathbb{C}^3$. Note that the economic resolution Y of $X = \mathbb{C}^3/G$ can be obtained by the sequence of the weighted blowups:

$$Y \xrightarrow{\varphi_2} Y_1 \xrightarrow{\varphi_1} X,$$

where φ_1 is the weighted blowup with weight $(1,2,1)$ and φ_2 is the toric morphism induced by the weighted blowup with weight $(2,1,2)$. The fan corresponding to Y consists of the following five maximal cones and their faces:

$$\sigma_1 = \mathrm{Cone}(e_1, e_3, v_2), \quad \sigma_2 = \mathrm{Cone}(e_1, v_2, v_1), \quad \sigma_2 = \mathrm{Cone}(e_1, v_1, e_2),$$
$$\sigma_4 = \mathrm{Cone}(e_3, e_2, v_1), \quad \sigma_5 = \mathrm{Cone}(e_3, v_1, v_2).$$

The following

$$\Gamma_1 = \{1, \eta, \eta^2\}, \quad \Gamma_2 = \{1, \eta, \zeta\}, \quad \Gamma_3 = \{1, \zeta, \zeta^2\},$$
$$\Gamma_4 = \{1, \xi, \xi^2\}, \quad \Gamma_5 = \{1, \xi, \eta\}.$$

are their corresponding G-iraffes. ♦

4.1.2 G-iraffes for $\frac{1}{r}(1, a, r-a)$

In this section, we assign a G-iraffe Γ_σ for each full dimensional cone in the fan of Y with $S(\Gamma_\sigma) = \sigma^\vee \cap M$

Let X be the quotient variety $\mathbb{C}^3/G$ where $G \subset \mathrm{GL}_3(\mathbb{C})$ is the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. Let $\varphi\colon Y \to X$ be the economic resolution of X. Then Y can be covered by U_2, U_3 and U_1, which are the unions of the affine toric varieties corresponding to the cones on the left side of, the right side of, and below the vector $v = \frac{1}{r}(1, a, r-a)$, respectively.

Assume σ is a full dimensional cone in the fan of Y. We have three cases:

(1) the cone σ is below the vector v.

(2) the cone σ is on the left side of the vector v.

(3) the cone σ is on the right side of the vector v.

Case (1) the cone σ is below the vector v. This means that the toric cone σ is smooth and that the toric affine open set U_σ is equal to U_1. Then consider the central round down function ϕ_1 of the weighted blowup with weight $(1, a, r-a)$. Now, for $\mathbf{m} = x^{m_1}y^{m_2}z^{m_3} \in \overline{M}$

$$\phi_1(\mathbf{m}) = \mathbf{1} \quad \text{if and only if} \quad 0 \le m_1 \le r-1 \text{ and } m_2 = m_3 = 0.$$

Thus the set $\Gamma := \phi_1^{-1}(\mathbf{1}) = \{1, x, x^2, \dots, x^{r-1}\}$ is a G-graph with $S(\Gamma) = \sigma^\vee \cap M$. Since the corresponding cone $\sigma(\Gamma)$ of Γ is equal to σ, Γ is a G-iraffe.

Case (2) the cone σ is on the left side of v. Consider the left round down function ϕ_2. From the fan of the economic resolution, it follows that U_2 is isomorphic to the economic resolution Y_2 for the group $G_2 = \frac{1}{a}(1, -r, r)$ with eigencoordinates ξ, η, ζ. There exists a unique full dimensional cone σ' in the fan of Y_2.

Lemma 4.1.4. *Let σ be a full dimensional cone in the toric fan of Y on the left side of the lattice v and σ' the corresponding full dimensional cone in the fan of Y_2, where Y_2 is the economic resolution for the group $G_2 = \frac{1}{a}(1, -r, r)$. Assume that there exists a G_2-graph Γ' such that $S(\Gamma') = (\sigma')^\vee \cap M$. Define a set*

$$\Gamma := \left\{ \mathbf{m} \in \overline{M} \,\middle|\, \phi_2(\mathbf{m}) \in \Gamma' \right\}.$$

Then Γ is a G-graph.

Proof. Firstly note that $\mathbf{1} \in \Gamma$ since $\phi_2(\mathbf{1}) = \mathbf{1} \in \Gamma'$. To show that Γ satisfies the second condition in Definition 2.4.1, let $\rho \in G^\vee$ be an irreducible representation of G. We have to show that there exists a unique monomial of weight ρ in Γ. Then there exists a positive integer i such that the weight of x^i is ρ. Consider the monomial $\phi_2(x^j)$ in M_2 and its weight χ in terms of the G_2-action. Since Γ' is a G_2-graph, there exists a unique element $\mathbf{k}_\chi$ whose weight is the same as the weight of $\phi_2(x^j)$. Then $\left(\frac{\mathbf{k}_\chi}{\phi_2(x^j)}\right)$ is in the G_2-invariant monomial lattice M, so it is in the monomial lattice $\overline{M}$. From Remark 3.2.3, it follows that

$$\phi_2 \colon x^j \cdot \left(\frac{\mathbf{k}_\chi}{\phi_2(x^j)} \right) \longmapsto \mathbf{k}_\chi,$$

i.e. $x^j \cdot \left(\frac{\mathbf{k}_\chi}{\phi_2(x^j)} \right)$ is in Γ. To show uniqueness, assume that two Laurent monomials $\mathbf{m}, \mathbf{n}$ of the same weights are mapped into Γ'. From the fact that the weights of $\phi_2(\mathbf{m})$ and $\phi_2(\mathbf{n})$ are equal, it follows that $\phi_2(\mathbf{m}) = \phi_2(\mathbf{n})$. From Remark 3.2.3,

$$\phi_2(\mathbf{m}) = \phi_2 \left(\mathbf{n} \cdot \frac{\mathbf{m}}{\mathbf{n}} \right) = \phi_2(\mathbf{n}) \cdot \frac{\mathbf{m}}{\mathbf{n}},$$

and hence $\mathbf{m} = \mathbf{n}$.

Lastly, to show Γ is connected, let $\mathbf{m} = x^{m_1} y^{m_2} z^{m_3} \in \overline{M}$ be an arbitrary element in Γ, i.e. $\mathbf{k}_\chi := \phi_2(\mathbf{m}) \in \Gamma'$. Consider the following six cases:

(A) Suppose $\xi \cdot \mathbf{k}_\chi$ is in Γ', but $\xi \cdot \mathbf{k}_\chi \neq \phi_2(x \cdot \mathbf{m})$. This means that

$$\frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3 \geq \left\lfloor \frac{1}{r} m_1 + \frac{a}{r} m_2 + \frac{r-a}{r} m_3 \right\rfloor + \frac{r-1}{r}.$$

From this equation, it is easy to show that $\phi_2(\frac{\mathbf{m}}{y}) = \mathbf{k}_\chi$ and $\phi_2(x \cdot \frac{\mathbf{m}}{y}) = \xi \cdot \mathbf{k}_\chi$. Hence, we can see that there is a path from $\mathbf{m}$ to $x \cdot \frac{\mathbf{m}}{y}$ in Γ and that $\phi_2(x \cdot \frac{\mathbf{m}}{y}) = \xi \cdot \mathbf{k}_\chi$.

(B) Suppose $\frac{\mathbf{k}_\chi}{\xi}$ is in Γ', but $\frac{\mathbf{k}_\chi}{\xi} \neq \phi_2(\frac{\mathbf{m}}{x})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 + \frac{r-1}{r} < \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor .$$

From this equation, it is easy to see that $\phi_2(y \cdot \mathbf{m}) = \mathbf{k}_\chi$ and $\phi_2(\frac{y \cdot \mathbf{m}}{x}) = \frac{\mathbf{k}_\chi}{\xi}$. Hence, there is a path from $\mathbf{m}$ to $\frac{y \cdot \mathbf{m}}{x}$ in Γ and $\phi_2(\frac{y \cdot \mathbf{m}}{x}) = \frac{\mathbf{k}_\chi}{\xi}$.

(C) Suppose $\eta \cdot \mathbf{k}_\chi$ is in Γ', but $\eta \cdot \mathbf{k}_\chi \neq \phi_2(y \cdot \mathbf{m})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{r-a}{r} < \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor .$$

From this, it is easy to show that there exists a positive integer k_0 such that $\phi_2(y^k \cdot \mathbf{m}) = \phi_2(\mathbf{m}) = \mathbf{k}_\chi$ for all $0 \leq k \leq k_0$ and $\phi_2(y^{k_0+1} \cdot \mathbf{m}) = \eta \cdot \mathbf{k}_\chi$. Hence, we can see that there is a path from $\mathbf{m}$ to $y^{k_0+1} \cdot \mathbf{m}$ in Γ and we get $\phi_2\big(y^{k_0+1} \cdot \mathbf{m}\big) = \eta \cdot \mathbf{k}_\chi$.

(D) Suppose $\frac{\mathbf{k}_\chi}{\eta}$ is in Γ', but $\frac{\mathbf{k}_\chi}{\eta} \neq \phi_2(\frac{\mathbf{m}}{y})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{a}{r} \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor .$$

From this, it is easy to see that there exists a positive integer k_0 such that $\phi_2(\frac{\mathbf{m}}{y^k}) = \phi_2(\mathbf{m}) = \mathbf{k}_\chi$ for all $0 \leq k \leq k_0$ and $\phi_2\Big(\frac{\mathbf{m}}{y^{k_0+1}}\Big) = \frac{\mathbf{k}_\chi}{\eta}$. Hence, there is a path from $\mathbf{m}$ to $\frac{\mathbf{m}}{y^{k_0+1}}$ in Γ and $\phi_2\Big(\frac{\mathbf{m}}{y^{k_0+1}}\Big) = \frac{\mathbf{k}_\chi}{\eta}$.

(E) Suppose $\zeta \cdot \mathbf{k}_\chi$ is in Γ', but $\zeta \cdot \mathbf{k}_\chi \neq \phi_2(z \cdot \mathbf{m})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{a}{r} \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor .$$

From this, it is easy to see that there exists a positive integer k_0[4] such that $\phi_2(\frac{\mathbf{m}}{y^k}) = \phi_2(\mathbf{m}) = \mathbf{k}_\chi$ for all $0 \leq k \leq k_0$ and $\phi_2\Big(\frac{\mathbf{m}}{y^{k_0+1}}\Big) \neq \mathbf{k}_\chi$. Moreover, $\phi_2(z \cdot \frac{\mathbf{m}}{y^{k_0}}) = \zeta \cdot \mathbf{k}_\chi$. Hence, there is a path from $\mathbf{m}$ to $z \cdot \frac{\mathbf{m}}{y^{k_0}}$ in Γ and

[4] This integer k_0 is the maximal integer satisfying

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{a}{r}k \geq \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor .$$

$\phi_2(z \cdot \frac{\mathbf{m}}{y^{k_0}}) = \zeta \cdot \mathbf{k}_\chi$.

(F) Suppose $\frac{\mathbf{k}_\chi}{\zeta}$ is in Γ', but $\frac{\mathbf{k}_\chi}{\zeta} \neq \phi_2(\frac{\mathbf{m}}{z})$. This means that

$$\frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 - \frac{r-a}{r} < \left\lfloor \frac{1}{r}m_1 + \frac{a}{r}m_2 + \frac{r-a}{r}m_3 \right\rfloor.$$

From this, it is easy to see that there exists a positive integer k_0 such that $\phi_2(y^k \cdot \mathbf{m}) = \phi_2(\mathbf{m}) = \mathbf{k}_\chi$ for all $0 \leq k \leq k_0$ and $\phi_2\left(y^{k_0+1} \cdot \mathbf{m}\right) \neq \mathbf{k}_\chi$. Moreover, $\phi_2(\frac{y^{k_0} \cdot \mathbf{m}}{z}) = \frac{\mathbf{k}_\chi}{\zeta}$. From this, it follows that there is a path from $\mathbf{m}$ to $\frac{y^{k_0} \cdot \mathbf{m}}{z}$ in Γ and that $\phi_2(\frac{y^{k_0} \cdot \mathbf{m}}{z}) = \frac{\mathbf{k}_\chi}{\zeta}$.

□

In proving Lemma 4.1.4, we have also proved the following lemma.

Lemma 4.1.5. *With the notation as above, for a monomial $\mathbf{k} \in \{\xi, \eta, \zeta\}$ of degree 1 and any $\mathbf{k}_\chi \in \Gamma'$, there exist a monomial $\mathbf{f} \in \{x, y, z\}$ of degree 1 and an element $\mathbf{m}_\rho \in \Gamma$ such that*

$$\phi_2(\mathbf{m} \cdot \mathbf{m}_\rho) = \mathbf{k} \cdot \mathbf{k}_\chi$$

with $\phi_2(\mathbf{m}_\rho) = \mathbf{k}_\chi$.

From Remark 3.2.3, it can be shown that

$$\mathrm{wt}_{\Gamma'}\left(\phi_2(\mathbf{m} \cdot \mathbf{m}_\rho)\right) = \phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)\big),$$

as they are elements in Γ' of the same weight.

Remark 4.1.6. By Lemma 3.2.5, it can be seen that if a Laurent monomial $\mathbf{m}_\rho$ of weight j is in Γ with $0 \leq j < r - a$, then $y \cdot \mathbf{m}_\rho$ is in Γ. ♦

Proposition 4.1.7. *With notation and assumptions as for Lemma 4.1.4, for the G-graph Γ, we have $S(\Gamma) = S(\Gamma')$. In particular, Γ is a G-iraffe with $S(\Gamma) = \sigma^\vee \cap M$.*

Proof. Note that $S(\Gamma)$ is generated by $\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}$ for $\mathbf{m} \in \overline{M}_{\geq 0}$ and $\mathbf{m}_\rho \in \Gamma$. Let $\mathbf{m}$ be a genuine monomial in $\overline{M}_{\geq 0}$ and $\mathbf{m}_\rho$ an element in Γ. From the definition of Γ, it follows that $\phi_2(\mathbf{m}_\rho)$ is in Γ', which is denoted by $\mathbf{k}_\chi \in \Gamma'$. Set $\mathbf{k}$ to be $\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathbf{m}_\rho}$. It is easy to see that $\mathbf{k}$ is a genuine monomial in ξ, η, ζ because of the definition of the left round down function. Since $\frac{\mathbf{m} \cdot \mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m} \cdot \mathbf{m}_\rho)}$ is G-invariant, from Remark 3.2.3, we have

$$\frac{\mathbf{m}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)} = \frac{\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)}{\phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)\big)} = \frac{\frac{\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)}{\phi_2(\mathbf{m}_\rho)}\cdot\phi_2(\mathbf{m}_\rho)}{\phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)\big)} = \frac{\mathbf{k}\cdot\mathbf{k}_\chi}{\mathrm{wt}_{\Gamma'}(\mathbf{k}\cdot\mathbf{k}_\chi)},$$

so we prove $S(\Gamma)\subset S(\Gamma')$. For the reverse inclusion, let $\frac{\mathbf{k}\cdot\mathbf{k}_\chi}{\mathrm{wt}_{\Gamma'}(\mathbf{k}\cdot\mathbf{k}_\chi)}$ be a generator of $S(\Gamma')$ with $\mathbf{k}\in\{\xi,\eta,\zeta\}$. It is sufficient to show that this generator is in $S(\Gamma)$. From Lemma 4.1.5, we can find $\mathbf{m}\in\overline{M}_{\geq 0}$ and $\mathbf{m}_\rho\in\Gamma$ satisfying $\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)=\mathbf{k}\cdot\mathbf{k}_\chi$. Note that $\mathrm{wt}_{\Gamma'}(\mathbf{k}\cdot\mathbf{k}_\chi)=\phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)\big)$. Thus we have

$$\frac{\mathbf{k}\cdot\mathbf{k}_\chi}{\mathrm{wt}_{\Gamma'}(\mathbf{k}\cdot\mathbf{k}_\chi)} = \frac{\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)}{\mathrm{wt}_{\Gamma'}\left(\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)\right)} = \frac{\phi_2(\mathbf{m}\cdot\mathbf{m}_\rho)}{\phi_2\big(\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)\big)} = \frac{\mathbf{m}\cdot\mathbf{m}_\rho}{\mathrm{wt}_\Gamma(\mathbf{m}\cdot\mathbf{m}_\rho)},$$

and we proved the proposition. □

Case (3) the cone σ is on the right side of v. We can get a similar result.

Corollary 4.1.8. *Let $G\subset\mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1,a,r-a)$ with a coprime to r. Let $\Sigma_{\max}$ be the set of 3-dimensional cones in the fan of the economic resolution Y of $X=\mathbb{C}^3/G$. Then there exists a set $\mathfrak{G}$ of G-iraffes such that there is a bijective map $\Sigma_{\max}\to\mathfrak{G}$ sending σ to Γ_σ satisfying $S(\Gamma_\sigma)=\sigma^\vee\cap M$. In particular, $U(\Gamma)$ is smooth for $\Gamma\in\mathfrak{G}$.*

Proof. From Section 4.1.1, note that the assertion holds when $a=1$ or $r-1$. We use induction on r and a.

Let $\Sigma_{\max}$ be the set of 3-dimensional cones in the fan of the economic resolution Y of $X=\mathbb{C}^3/G$ and σ an arbitrary element of $\Sigma_{\max}$. Then σ is either on the left side of the lattice $v=\frac{1}{r}(1,a,r-a)$, the right side of v, or below v.

For the case where σ is below v, define

$$\Gamma_\sigma := \{1,x,x^2,\dots,x^{r-2},x^{r-1}\}.$$

Then we have seen that Γ_σ is a G-iraffe with $S(\Gamma_\sigma)=\sigma^\vee\cap M$.

If the cone σ is on the left side of v, then we have a unique 3-dimensional cone σ' in the fan of the economic resolution of $\frac{1}{a}(1,\overline{-r},\overline{r})$ where $^{-}$ denotes the residue modulo a. Note that $\overline{-r}$ is strictly less than a. Using induction and Proposition 4.1.7, we prove that there exists a G-iraffe Γ_σ satisfying $S(\Gamma_\sigma)=\sigma^\vee\cap M$.

The case where the cone σ is on the right side of v can be proved similarly. □

Remark 4.1.9. Let Γ be a G-iraffe in $\mathfrak{G}$ and Γ' the corresponding G_k-graph i.e. $\Gamma' = \phi_k(\Gamma)$ with the round down function ϕ_k. As in Section 2.5, note that we have the affine set $D(\Gamma)$ through $C(\Gamma)$ whose coordinate ring is

$$\mathbb{C}\left[x_\rho, y_\rho, z_\rho \,\middle|\, \rho \in G^\vee\right] / I_\Gamma$$

where $I_\Gamma = \left\langle \text{ the quadrics in (2.5.8), } \mathbf{f} - 1 \,\middle|\, \mathbf{f} \in \Lambda(\Gamma) \right\rangle$. Let $D(\Gamma')$ be the affine open set $D(\Gamma')$ through $C(\Gamma')$. Similarly, the coordinate ring of $D(\Gamma')$ is

$$\mathbb{C}\left[\xi_\chi, \eta_\chi, \zeta_\chi \,\middle|\, \chi \in G_k^\vee\right] / I_{\Gamma'}$$

where $I_{\Gamma'} = \left\langle \text{ the quadrics in (2.5.8), } \mathbf{k} - 1 \,\middle|\, \mathbf{k} \in \Lambda(\Gamma') \right\rangle$.

We can prove that $D(\Gamma)$ and $D(\Gamma')$ are isomorphic by showing the following algebra homomorphism is an isomorphism between the coordinate rings of $D(\Gamma)$ and $D(\Gamma')$:

$$\mathbb{C}[D(\Gamma)] \to \mathbb{C}[D(\Gamma')] \quad \text{given by } \mathbf{f}_\rho \mapsto \mathbf{k}_\chi,$$

where $\mathbf{k} = \frac{\phi_k(\mathbf{f}\cdot\mathbf{m}_\rho)}{\phi_k(\mathbf{m}_\rho)}$, $\chi = \phi_k(\rho)$ and $\mathbf{m}_\rho$ is a unique element of weight ρ in Γ. Therefore we have the following diagrams:

$$\begin{tikzcd} U(\Gamma') \arrow[r, hook, "\text{closed}"'] \arrow[d, "\cong"] & D(\Gamma') \arrow[d, "\cong"] \\ U(\Gamma) \arrow[r, hook, "\text{closed}"'] & D(\Gamma'), \end{tikzcd} \qquad \begin{tikzcd} \mathbb{C}[D(\Gamma')] \arrow[r, two heads, "\text{surj.}"'] \arrow[d, "\cong"] & \mathbb{C}[S(\Gamma')] \arrow[d, "\cong"] \\ \mathbb{C}[D(\Gamma')] \arrow[r, two heads, "\text{surj.}"'] & \mathbb{C}[S(\Gamma')]. \end{tikzcd}$$

If $a = 1$ or $r - 1$, then $U(\Gamma) = D(\Gamma)$ for $\Gamma \in \mathfrak{G}$ as is is G-Hilb and irreducible by Kędzierski [15] (see Theorem 4.1.1). By induction on a and r, we can proves that $U(\Gamma) = D(\Gamma)$ for $\Gamma \in \mathfrak{G}$. ♦

Example 4.1.10. Let G be the group of type $\frac{1}{7}(1,3,4)$ as in Example 2.4.3. The fan of the economic resolution of the quotient variety is shown in Figure 3.3.1.

Let us define the following cones:

$$\sigma_1 := \text{Cone}\left((1,0,0), \tfrac{1}{7}(1,3,4), \tfrac{1}{7}(3,2,5)\right),$$

$$\sigma_2 := \text{Cone}\left((1,0,0), \tfrac{1}{7}(6,4,3), \tfrac{1}{7}(1,3,4)\right).$$

We now calculate G-graphs associated to the cones σ_1 and σ_2. Note that the left side of the fan is the economic resolution of the quotient variety $\frac{1}{3}(1,2,1)$ which is G_2-Hilb $\mathbb{C}^3$, where G_2 is of type $\frac{1}{3}(1,2,1)$. Call the eigencoordinates ξ, η, ζ. Let σ_1'

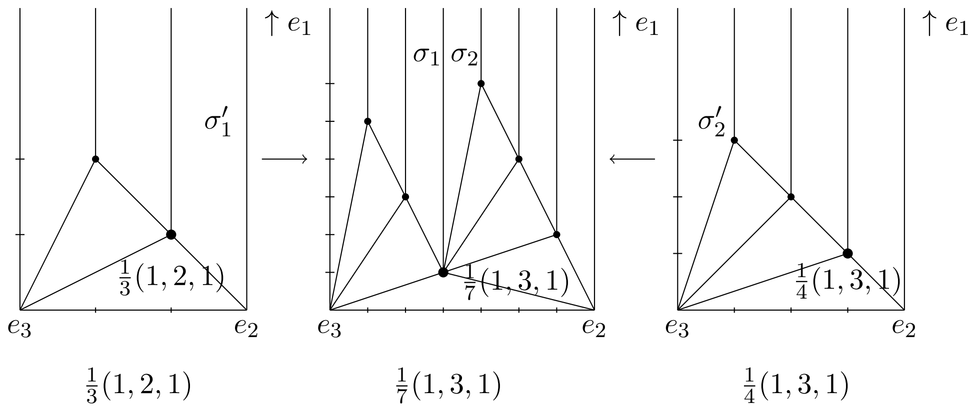


Figure 4.1.1: Recursion process for $\frac{1}{7}(1,3,4)$

be the cone in the fan of G_2-Hilb $\mathbb{C}^3$ which corresponds to σ_1. Observe that the corresponding G_2-graph Γ'_1 is

$$\Gamma'_1 = \{1, \zeta, \zeta^2\},$$

and that the left round down function ϕ_2 is

$$\phi_2(x^{m_1}y^{m_2}z^{m_3}) = \xi^{m_1}\eta^{\lfloor \frac{1}{7}m_1 + \frac{3}{7}m_2 + \frac{4}{7}m_3 \rfloor}\zeta^{m_3}.$$

Thus G-graph Γ_1 corresponding to σ_1 is

$$\begin{aligned}\Gamma_1 &\stackrel{\text{def}}{=} \{x^{m_1}y^{m_2}z^{m_3} \in \overline{M} \mid \phi_2(x^{m_1}y^{m_2}z^{m_3}) \in \Gamma'_1\} \\ &= \{1, y, y^2, z, \tfrac{z}{y}, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}\}.\end{aligned}$$

For the cone σ_2, note that the right side of the fan is the economic resolution of the quotient variety $\frac{1}{4}(1,3,1)$ which is G_3-Hilb $\mathbb{C}^3$, where G_3 is of type $\frac{1}{4}(1,3,1)$. Call the eigencoordinates α, β, γ. Let σ'_2 be the cone in the fan of G_2-Hilb $\mathbb{C}^3$ which corresponds to σ_2. Observe that the corresponding G_3-graph Γ'_2 is

$$\Gamma'_2 = \{1, \beta, \beta^2, \beta^3\},$$

and that the right round down function ϕ_3 is

$$\phi_3(x^{m_1}y^{m_2}z^{m_3}) = \alpha^{m_1}\beta^{m_2}\gamma^{\lfloor \frac{1}{7}m_1 + \frac{3}{7}m_2 + \frac{4}{7}m_3 \rfloor}.$$

Thus the G-graph Γ_2 corresponding to σ_2 is

$$\begin{aligned}\Gamma_2 &\stackrel{\text{def}}{=} \left\{x^{m_1}y^{m_2}z^{m_3} \in \overline{M} \,\middle|\, \phi_2(x^{m_1}y^{m_2}z^{m_3}) \in \Gamma'_2\right\} \\ &= \left\{1, z, y, y^2, \tfrac{y^2}{z}, \tfrac{y^3}{z^1}, \tfrac{y^3}{z^2}\right\}.\end{aligned}$$

From Example 2.4.6, $\sigma(\Gamma_1) = \sigma_1$ and $\sigma(\Gamma_2) = \sigma_2$. ♦

4.2 A chamber in the stability parameter space

This section proves that there exists a chamber $\mathfrak{C}$ such that the admissible G-iraffes in Section 4.1 are θ-stable for $\theta \in \mathfrak{C}$. In addition, we prove that the chamber $\mathfrak{C}$ coincides with the cone Kędzierski found and that the chamber is an open Weyl chamber. Moreover, it turns out that this chamber is a full chamber, i.e. the facets of $\mathfrak{C}$ form actually walls (see Section 5.2).

4.2.1 Admissible chambers

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. We may assume $2a < r$. Let G_2 and G_3 be the groups of type $\frac{1}{a}(1, \overline{-r}, \overline{r})$ and of type $\frac{1}{r-a}(1, \overline{r}, \overline{-r})$, respectively. Note that for $k = 2$ or 3, the round down function ϕ_k induces a surjection $\phi_k \colon G^\vee \to G_k^\vee$.

The stability parameter space for G_k-constellations is

$$\Theta_k = \left\{\theta \in \mathrm{Hom}_{\mathbb{Z}}\left(R(G_k), \mathbb{Q}\right) \,\middle|\, \theta\big(\mathbb{C}[G_k]\big) = 0\right\}$$

where $R(G_k)$ is the representation ring of G_k, i.e. $R(G_k) = \bigoplus_{\chi \in G_k^\vee} \mathbb{Z}\chi$. Let us assume that there exists a stability parameter $\theta^{(k)} \in \Theta_k$ such that the admissible G_k-graphs are $\theta^{(k)}$-stable. Take a GIT parameter $\theta_P \in \Theta$ satisfying the following system of linear equations:

$$\begin{cases}\theta^{(2)}(\chi) = \theta\big(\phi_2^{-1}(\chi)\big) & \text{for all } \chi \in G_2^\vee, \\ \theta^{(3)}(\chi') = \theta\big(\phi_3^{-1}(\chi')\big) & \text{for all } \chi' \in G_3^\vee.\end{cases} \tag{4.2.1}$$

Let us define a GIT parameter $\vartheta \in \Theta$ to be

$$\vartheta(\rho) = \begin{cases}-1 & \text{if } 0 \le \mathrm{wt}(\rho) < a\ , \\ 1 & \text{if } r-a \le \mathrm{wt}(\rho) < r, \\ 0 & \text{otherwise.}\end{cases} \tag{4.2.2}$$

Note that $\vartheta\big(\phi_k^{-1}(\chi)\big) = 0$ for any $\chi \in G_k^{\vee}$ [5]. For a sufficiently large natural number m, set
$$\theta := \theta_P + m\vartheta. \tag{4.2.3}$$
We claim that the admissible G-iraffes are θ-stable.

Lemma 4.2.4. *Let θ be the parameter as above. For the set $\mathfrak{G}$ in Corollary 4.1.8, if Γ is in $\mathfrak{G}$, then Γ is θ-stable.*

Proof. Let Γ be a G-iraffe in $\mathfrak{G}$ and σ the corresponding cone to Γ. It suffices to show that $C(\Gamma)$ is θ-stable. We have three cases as in Section 4.1.2:

(1) the cone σ is below the vector v.

(2) the cone σ is on the left side of the vector v.

(3) the cone σ is on the right side of the vector v.

In Case (1), we have only one G-iraffe
$$\Gamma = \{1, x, x^2, \ldots, x^{r-2}, x^{r-1}\}.$$
By Lemma 2.5.1, any nonzero proper submodule $\mathcal{G}$ of $C(\Gamma)$ is given by the set
$$A = \{x^j, x^{j+1}, \ldots, x^{r-2}, x^{r-1}\}$$
for some $1 \leq j \leq r-1$. Since m is sufficiently large, it follows that $\theta(\mathcal{G}) > 0$ so Γ is θ-stable.

We now prove the result in Case (2).

Let Γ be a G-iraffe with corresponding G_2-graph Γ'. Let $\mathcal{G}$ be a submodule of $C(\Gamma)$ whose $\mathbb{C}$-basis is $A \subset \Gamma$. Remark 4.1.6 and Lemma 2.5.1 imply that if $\mathbf{m}_\rho \in A$ for $0 \leq \mathrm{wt}(\mathbf{m}_\rho) < a$, then $\phi_2^{-1}\big(\phi_2(\mathbf{m}_\rho)\big) \subset A$. Thus $\vartheta(\mathcal{G}) \geq 0$ from the definition of θ as m is sufficiently large.

If $\vartheta(\mathcal{G}) > 0$, then since m is sufficiently large, it follows that $\theta(\mathcal{G}) > 0$.

If $\vartheta(\mathcal{G}) = 0$, then one can see that $A = \phi_2^{-1}\big(\phi_2(A)\big)$. Let us assume that $A = \phi_2^{-1}\big(\phi_2(A)\big)$. To show this, we prove that $\phi_2(A)$ gives a submodule $\mathcal{G}'$ of $C(\Gamma')$ and that $\theta(\mathcal{G}) = \theta^{(2)}(\mathcal{G}')$. Since θ satisfies the system of linear equations (4.2.1), it suffices to show that $\phi_2(A)$ gives a submodule $\mathcal{G}'$ of $C(\Gamma')$. Recall ξ, η, ζ are the coordinates of $\mathbb{C}^3$ with respect to the action of G_2. By Lemma 2.5.1, it is enough to show that if $\mathbf{k} \cdot \phi_2(\mathbf{m}_\rho) \in \Gamma'$, then $\mathbf{k} \cdot \phi_2(\mathbf{m}_\rho)$ in $\phi_2(A)$ for any $\mathbf{k} \in \{\xi, \eta, \zeta\}$ and

[5] One can see if any $\theta \in \Theta$ satisfies that $\theta\big(\phi_k^{-1}(\chi)\big) = 0$ for any $\chi \in G_k^{\vee}$, then θ must be a constant multiple of ϑ. This also explains the existence of a solution θ_P for (4.2.1).

$\mathbf{m}_\rho \in A$. Suppose $\mathbf{k} \cdot \phi_2(\mathbf{m}_\rho) \in \Gamma'$ for some $\mathbf{m}_\rho \in A$. By Lemma 4.1.5, there exists $\mathbf{m}_{\rho'}$ such that

$$\phi_2(\mathbf{f} \cdot \mathbf{m}_{\rho'}) = \mathbf{k} \cdot \mathbf{k}_\chi$$

with $\phi_2(\mathbf{m}_{\rho'}) = \phi_2(\mathbf{m}_\rho)$ for some $\mathbf{f} \in \{x, y, z\}$. In particular, $\mathbf{f} \cdot \mathbf{m}_{\rho'} \in \Gamma = \phi_2^{-1}(\Gamma')$. Since $A = \phi_2^{-1}\big(\phi_2(A)\big)$, we have $\mathbf{m}_{\rho'} \in A$, which implies $\mathbf{f} \cdot \mathbf{m}_{\rho'} \in A$ as A is a $\mathbb{C}$-basis of $\mathcal{G}$. Thus $\mathbf{k} \cdot \mathbf{k}_\chi$ is in $\phi_2(A)$. □

4.2.2 Root system in A_{r-1}

We review well known facts on the A_{r-1} root system. Let $I := \mathrm{Irr}(G)$ be identified with $\mathbb{Z}/r\mathbb{Z}$. As is well known, the following three are in 1-to-1 correspondence:

(1) Sets of simple roots Δ.

(2) Open Weyl Chambers $\mathfrak{C}$.

(3) Elements of $S_r := \big\{\omega \,\big|\, \omega \text{ is a permutation of } I\big\}$.

Let $\big\{\varepsilon_i \,\big|\, i \in I\big\}$ be an orthonormal basis of $\mathbb{Q}^r$, i.e. $\langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$. Note that the indices are in $I = \mathbb{Z}/r\mathbb{Z}$. Define

$$\Phi := \big\{\varepsilon_i - \varepsilon_j \,\big|\, i, j \in I, i \neq j\big\}.$$

Let $\mathfrak{h}^*$ be the subspace of $\mathbb{Q}^r$ generated by Φ. Elements in Φ are called *roots*. For each nonzero $i \in I$, set $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$. For any root α, one can see that $\langle \alpha, \alpha \rangle = 2$. Note that

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } |i-j| = a, \\ 0 & \text{otherwise.} \end{cases}$$

This is the root system of A_{r-1} and the Weyl group of this root system is the group generated by simple reflections

$$s_i \colon \alpha \mapsto \alpha - \frac{\langle \alpha, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i.$$

It is easy to see that

$$s_i(\varepsilon_k - \varepsilon_l) = \varepsilon_{\omega_i(k)} - \varepsilon_{\omega_i(l)},$$

where ω_i is the (adjacent) transposition in S_r

$$\omega_i(j) = \begin{cases} i+a & \text{if } j = i, \\ i & \text{if } j = i+a, \\ j & \text{otherwise.} \end{cases}$$

Thus the Weyl group can be thought as the group of permutations of I.

Here, we consider roots as dimension vectors:

(i) α_i is the dimension vector of the vertex simple at the vertex ρ_i;

(ii) the dimension vector of the vertex simple at the trivial representation ρ_0 is $-\sum_{i\neq 0} \alpha_i$.

The stability parameter space Θ can be identified with the dual space of $\mathfrak{h}^*$. Let ω be a permutation of I. As is customary (see e.g. [11]), define a set of simple roots and an open Weyl chamber associated to ω:

$$\begin{aligned} \Delta(\omega) &:= \left\{\varepsilon_{\omega(i)} - \varepsilon_{\omega(i-a)} \in \Phi \,\middle|\, i \in I, i \neq 0\right\}, \\ \mathfrak{C}(\omega) &:= \left\{\theta \in (\mathfrak{h}^*)^* \,\middle|\, \theta\big(\varepsilon_{\omega(i)} - \varepsilon_{\omega(i-a)}\big) > 0 \quad \forall i \in I, i \neq 0\right\}. \end{aligned}$$

In particular, for the identity permutation of I, the corresponding simple roots Δ_+ and Weyl chamber $\mathfrak{C}_+$ are

$$\begin{aligned} \Delta_+ &= \left\{\varepsilon_i - \varepsilon_{i-a} \in \Phi \,\middle|\, i \in I, i \neq 0\right\} = \left\{\alpha_i \,\middle|\, i \in I, i \neq 0\right\}, \\ \mathfrak{C}_+ &= \left\{\theta \in (\mathfrak{h}^*)^* \,\middle|\, \theta\big(\alpha_i\big) > 0 \quad \forall i \in I, i \neq 0\right\}, \end{aligned}$$

which is the cone Θ_+ for G-Hilb in (2.2.10).

A chamber in stability parameter space. For each $i \in I$, let ρ_i denote the irreducible representation of G of weight i. Note that each root α can be considered as the support of a submodule of a G-constellation. In other words, α_i corresponds to the dimension vector of ρ_i. Thus in general root $\alpha = \sum_i n_i\alpha_i$ is the dimension vector of the representation $\oplus n_i\rho_i$. Abusing notation, let $\alpha = \sum_i n_i\alpha_i$ also denote the corresponding representation $\oplus n_i\rho_i$.

Let Δ be a set of simple roots. Define a subset $\mathfrak{C}$ of Θ associated to Δ as

$$\mathfrak{C} := \mathfrak{C}(\Delta) := \left\{\theta \in \Theta \,\middle|\, \theta(\alpha) > 0 \quad \forall \alpha \in \Delta\right\}.$$

At this moment, $\mathfrak{C}(\Delta)$ is not necessarily a chamber in Θ because $\mathfrak{C}(\Delta)$ may contain nongeneric elements.

4.2.3 Admissible sets of simple roots

In this section, we define the *admissible set of simple roots* Δ_a for the group of type $\frac{1}{r}(1, a, r-a)$. The Weyl chamber $\mathfrak{C}_a$ corresponding to the admissible set of simple roots is equal to the GIT stability parameter cone in [16].

Remark 4.2.5. Kędzierski [16] described a cone of GIT parameters with a set of inequalities. One can easily see that this can be described using the root system A_{r-1}. He conjectured the cone is a full chamber. In Section 5.2, we prove that the conjecture is true. ♦

Firstly, we consider the case of $\frac{1}{r}(1, r-1, 1)$. Secondly, we define the admissible set of simple roots for $\frac{1}{r}(1, a, r-a)$ using a recursion process.

The case of $\frac{1}{r}(1, r-1, 1)$. From Theorem 4.1.1, we know that the economic resolution of the quotient variety $X = \mathbb{C}^3/G$ is isomorphic to G-Hilb $\mathbb{C}^3$ where G is of type $\frac{1}{r}(1, r-1, 1)$. Thus in this case, the G-iraffes are just Nakamura G-graphs which are θ-stable for $\theta \in \Theta_+$, where

$$\Theta_+ := \left\{\theta \in \Theta \,\middle|\, \theta(\rho) > 0 \text{ for } \rho \neq \rho_0\right\}.$$

In terms of the root system, $\theta(\alpha_i) > 0$ for nonzero $i \in I$. Note that $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$. Thus the corresponding set of simple roots is

$$\Delta = \left\{\varepsilon_i - \varepsilon_{i-a} \in \Phi \,\middle|\, i \in I, i \neq 0\right\}.$$

Example 4.2.6. Consider the group of type $\frac{1}{3}(1, 2, 1)$. Let $\left\{\varepsilon_j^L \,\middle|\, j = 0, 1, 2\right\}$ be the standard basis of $\mathbb{Q}^3$. Then the corresponding set of simple roots Δ^L is

$$\Delta^L = \left\{\varepsilon_1^L - \varepsilon_2^L, \varepsilon_2^L - \varepsilon_0^L\right\}.$$

On the other hand, for the group of type $\frac{1}{4}(1, 3, 1)$, let $\left\{\varepsilon_k^R \,\middle|\, k = 0, 1, 2, 3\right\}$ be the standard basis of $\mathbb{Q}^4$. Then

$$\Delta^R = \{\varepsilon_1^R - \varepsilon_2^R, \varepsilon_2^R - \varepsilon_3^R, \varepsilon_3^R - \varepsilon_0^R\}$$

is the corresponding set of simple roots for type $\frac{1}{4}(1, 3, 1)$. ♦

The case of $\frac{1}{r}(1,a,r-a)$. Let G be the group of type $\frac{1}{r}(1,a,r-a)$. Let us assume that for $\frac{1}{a}(1,\overline{-r},\overline{r})$ and $\frac{1}{r-a}(1,\overline{a},\overline{-r})$ we have sets of simple roots Δ^L and Δ^R, respectively. Note that Δ^L is a set of simple roots in A_{a-1} and Δ^R is a set of simple roots in A_{r-a-1}. As in Section 4.2.2, let

$$\{\varepsilon_l^L \mid l=0,1,\dots,a-1\}, \quad \{\varepsilon_k^R \mid k=0,1,\dots,r-a-1\}$$

be the standard basis of $\mathbb{Q}^a$ and $\mathbb{Q}^{r-a}$, respectively. From the two sets of simple roots Δ^L and Δ^R, we construct a set Δ of simple roots in A_{r-1} as follows. Firstly, as in Section 4.2.2, let the standard basis $\{\varepsilon_i \mid i\in I\}$ of $\mathbb{Q}^r$ be identified with the union of the two sets

$$\{\varepsilon_l^L \mid l=0,1,\dots,a-1\} \text{ and } \{\varepsilon_k^R \mid k=0,1,\dots,r-a-1\}$$

using the following identification:

$$\begin{aligned} \varepsilon_l^L &= \varepsilon_i \quad \text{with } i\equiv l \mod a, && r-a\le i<r,\\ \varepsilon_k^R &= \varepsilon_i \quad \text{with } i\equiv k \mod (r-a), && 0\le i<r-a. \end{aligned} \tag{4.2.7}$$

Secondly, with this identification, define a set Δ of simple roots

$$\Delta = \Delta^L \cup \{\varepsilon_{\lfloor \frac{r}{a}\rfloor a} - \varepsilon_{(r-2a)-\lfloor \frac{r-2a}{r-a}\rfloor (r-a)}\} \cup \Delta^R. \tag{4.2.8}$$

Note that Δ is actually a set of simple roots in A_{r-1}.

Remark 4.2.9. Note that if $\varepsilon_l^L-\varepsilon_k^L$ is a positive sum of simple roots in Δ^L, then the corresponding root of A_{r-1} is also a positive sum of simple roots in Δ. Moreover, $\varepsilon_l^L-\varepsilon_k^R$ can be written as a positive sum of simple roots in Δ: note that $\varepsilon_{\lfloor \frac{r}{a}\rfloor a}$ is identified with a vector ε^L and that $\varepsilon_{(r-2a)-\lfloor \frac{r-2a}{r-a}\rfloor (r-a)}$ is identified with a vector ε^R; since we add the root $\varepsilon_{\lfloor \frac{r}{a}\rfloor a} - \varepsilon_{(r-2a)-\lfloor \frac{r-2a}{r-a}\rfloor (r-a)}$ to Δ, $\varepsilon_l^L-\varepsilon_k^R$ is a positive sum of simple roots in Δ. ♦

Definition 4.2.10. With the notation as above, we call the set Δ of simple roots the *admissible set of simple roots* for $G=\frac{1}{r}(1,a,r-a)$, which is denoted by Δ_a. For the admissible set of simple roots, define

$$\mathfrak{C}_a := \{\theta\in\Theta \mid \theta(\alpha)>0 \quad \forall\alpha\in\Delta_a\}$$

with considering roots $\alpha=\sum_i n_i\alpha_i$ as corresponding representations $\oplus n_i\rho_i$. We call $\mathfrak{C}_a$ the *admissible Weyl chamber* for $G=\frac{1}{r}(1,a,r-a)$.

As is stated in Section 4.2.2, note that a set of simple roots Δ_a is determined by and determines a permutation of $I = \mathbb{Z}/r\mathbb{Z}$. Indeed,

$$\Delta_a = \left\{ \varepsilon_{\omega(i)} - \varepsilon_{\omega(i-a)} \,\middle|\, i \in I, i \neq 0 \right\}$$

for a unique permutation $\omega \colon I \to I$.

Let $\{\theta_i\}_{i=1}^{r-1}$ be the dual basis of the GIT parameter space Θ with respect to $\{\alpha_i\}_{i=1}^{r-1}$, i.e. $\theta_i(\alpha_j) = \delta_{ij}$. Set $\theta_0 = -\sum_{i=1}^{r-1} \theta_i$. As is standard, we can present the rays of the Weyl chamber $\mathfrak{C}_a$ using this basis and the permutation ω: the rays are generated by the following vectors

$$\sum_{j=0}^{i-1} \left(\theta_{\omega(ja)+a} - \theta_{\omega(ja)} \right) \tag{4.2.11}$$

for $i = 1, 2, \ldots, r-1$. Thus any $\theta \in \mathfrak{C}_a$ is a positive linear sum of the vectors above in (4.2.11).

Proposition 4.2.12. *Assume that $a < r - a$. Let θ be an element in the admissible chamber $\mathfrak{C}_a$. Then $\theta(\alpha_i)$ is negative if and only if $0 \leq i < a$. Therefore any θ-stable G-constellation is generated by $\rho_0, \rho_1, \ldots, \rho_{a-1}$.*

Proof. Let $\theta \in \mathfrak{C}_a$. Recall that any root can be written as a sum of simple roots and that elements in Δ_a are positive on θ.

Suppose that $0 \leq i < a$. From the identification (4.2.7), one can see that ε_i is identified with ε_k^R for some l and that ε_{i-a} is identified with ε_l^L for some l. By Remark 4.2.9, the root $\alpha_i = \varepsilon_i - \varepsilon_{i-a} = \varepsilon^R - \varepsilon^L$ is a negative sum of simple roots in Δ_a.

Suppose that $r - a \leq i < r$. Consider the root $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$. From the identification (4.2.7), one can see that ε_i is identified with ε_k^L for some l and that ε_{i-a} is identified with ε_l^R for some l. Thus $\alpha_i = \varepsilon^L - \varepsilon^R$ is a positive sum of simple roots in Δ_a by Remark 4.2.9.

Consider the case where $a \leq i < r - a$. The root $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$ is a sum of simple roots in Δ^R. A recursive argument yields that α_i is a positive sum of simple roots in Δ^R. Thus α_i is a positive sum of simple roots in Δ_a by Remark 4.2.9.

To prove the last statement, let $\mathcal{F}$ be a θ-stable G-constellation. Consider the submodule $\mathcal{G}$ of $\mathcal{F}$ generated by $\rho_0, \rho_1, \ldots, \rho_{a-1}$. If $\mathcal{G} \neq \mathcal{F}$, then $\theta(\mathcal{G}) < 0$. Therefore we have $\mathcal{G} = \mathcal{F}$. $\square$

Example 4.2.13. Let G be the group of type $\frac{1}{7}(1, 3, 4)$. From the fan of the economic resolution of this case (see Example 3.3.4), the left and right sides are the economic resolutions of singularities of $\frac{1}{3}(1,2,1)$ and $\frac{1}{4}(1,3,1)$, respectively. By Example 4.2.6, we have two sets

$$\Delta^L = \{\varepsilon_1^L - \varepsilon_2^L, \varepsilon_2^L - \varepsilon_0^L\} \text{ and } \Delta^R = \{\varepsilon_1^R - \varepsilon_2^R, \varepsilon_2^R - \varepsilon_3^R, \varepsilon_3^R - \varepsilon_0^R\}.$$

As in the construction (4.2.8), the admissible set of simple roots is

$$\Delta_a = \{\varepsilon_4 - \varepsilon_5, \varepsilon_5 - \varepsilon_6, \underline{\varepsilon_6 - \varepsilon_1}, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_0\},$$

where the underlined root is the added root as in (4.2.8). In terms of $\alpha_i = \varepsilon_i - \varepsilon_{i-a}$,

$$\Delta_a = \{\alpha_4 + \alpha_1, \alpha_5 + \alpha_2, \underline{-\alpha_1 - \alpha_5 - \alpha_2}, \alpha_1 + \alpha_5, \alpha_2 + \alpha_6, \alpha_3\}.$$

Thus the set of parameters $\theta \in \Theta$ satisfying

$$\begin{array}{lll} \theta(\rho_4 \oplus \rho_1) > 0, & \theta(\rho_5 \oplus \rho_2) > 0, & \theta(\rho_1 \oplus \rho_5 \oplus \rho_2) < 0, \\ \theta(\rho_1 \oplus \rho_5) > 0, & \theta(\rho_2 \oplus \rho_6) > 0, & \qquad \theta(\rho_3) > 0 \end{array}$$

is the admissible Weyl chamber $\mathfrak{C}_a$ where ρ_i is the irreducible representation of G of weight i.

The corresponding permutation ω is

$$\omega = \begin{pmatrix} 0 & 3 & 6 & 2 & 5 & 1 & 4 \\ 0 & 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$$

i.e. $\omega(0) = 0$, $\omega(3) = 3$, $\omega(6) = 2$, etc. The rays of the Weyl chamber $\mathfrak{C}_a$ are the row vectors of the matrix

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

with the basis $\{\theta_i\}$. Note that for any $\theta \in \mathfrak{C}_a$, $\theta(\alpha_i)$ is negative if and only if $0 \le i < 3$. ♦

4.2.4 An open Weyl chamber

In this section, we prove that the stability parameters described in Section 4.2.1 form an open Weyl chamber. It follows that our stability parameters are the same as Kędzierski's in [16].

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. We may assume $2a < r$. Let G_2 and G_3 be the groups of type $\frac{1}{a}(1, \overline{-r}, \overline{r})$ and of type $\frac{1}{r-a}(1, \overline{r}, \overline{-r})$, respectively. To use recursion steps, assume that the admissible set of simple roots Δ^L and Δ^R give the full chambers $\mathfrak{C}^L$ and $\mathfrak{C}^R$. Let Δ_a be the admissible set of simple roots and $\mathfrak{C}_a$ the admissible Weyl chamber for $\frac{1}{r}(1, a, r-a)$.

We prove that $\mathfrak{C}_a$ is a full chamber such that the admissible G-iraffes are θ-stable for $\theta \in \mathfrak{C}_a$ by the following three steps.

Step 1 Firstly, we prove that for any $\theta \in \mathfrak{C}_a$, there exist $\theta^{(2)} \in \mathfrak{C}^L$ and $\theta^{(3)} \in \mathfrak{C}^R$ such that θ is a partial solution of the system of linear equations (4.2.1). Let θ be in $\mathfrak{C}_a$. Let us define $\theta^{(2)}$, $\theta^{(3)}$ to be

$$\begin{cases} \theta^{(2)}(\chi) = \theta\big(\phi_2^{-1}(\chi)\big) & \text{for } \chi \in G_2^{\vee}, \\ \theta^{(3)}(\chi') = \theta\big(\phi_3^{-1}(\chi')\big) & \text{for } \chi' \in G_3^{\vee}. \end{cases}$$

It suffices to show that $\theta^{(2)} \in \mathfrak{C}^L$ and $\theta^{(3)} \in \mathfrak{C}^R$. Let χ_l be a character of G_2 whose weight is l. Then

$$\phi_2^{-1}(\chi_l) = \left\{\rho_i \in G^{\vee} \,\middle|\, 0 \le i < r,\ i \equiv l \mod a\right\},$$

by the definition of the left round down function, so the dimension vector of $\phi_2^{-1}(\chi_l)$ in terms of roots is

$$\sum_{\substack{0 \le i < r, \\ i \equiv l \bmod a}} \alpha_i = \sum_{\substack{0 \le i < r, \\ i \equiv l \bmod a}} (\varepsilon_i - \varepsilon_{i-a}) = \alpha_l^L.$$

Note that θ is positive on Δ_a. In particular θ is positive on the roots coming from Δ^L. From this, it follows that $\theta^{(2)}$ is in $\mathfrak{C}^L$. For $\theta^{(3)}$, we can prove the assertion in a similar way.

Step 2 Secondly, we prove that the vector ϑ in (4.2.2) is a ray of the chamber $\mathfrak{C}_a$. From this, it follows that any $\theta \in \mathfrak{C}_a$ can be written as the form (4.2.3) so admissible G-iraffes are θ-stable.

Let ϑ be the vector in (4.2.2). As is well known, ϑ is a ray of the Weyl chamber $\mathfrak{C}_a$ associated to the set of simple root Δ_a if and only if there exists a unique simple root α in Δ_a such that $\vartheta(\alpha)$ is positive and ϑ is zero on the other simple roots in Δ_a. A simple observation shows that ϑ is zero on the sets Δ^L and Δ^R with the identification (4.2.8). It remains to show that $\vartheta(\alpha)$ is positive for

$$\begin{aligned}\alpha &= \varepsilon_{\lfloor \frac{r}{a} \rfloor a} - \varepsilon_{(r-2a)-\lfloor \frac{r-2a}{r-a} \rfloor (r-a)} = \varepsilon_{\lfloor \frac{r}{a} \rfloor a} - \varepsilon_{r-2a} \\ &= \sum_{\rho_i \in \phi_2^{-1}(A)} \alpha_i + \alpha_{r-a}.\end{aligned}$$

for a subset A of $G^\vee$. Since $\vartheta(A) = 0$ and $\vartheta(\alpha_{r-a}) = 1$, we have $\vartheta(\alpha) = 1$.

Step 3 Lastly, we prove that the chamber is a full chamber. By Step 1 and Step 2, we prove that the Weyl chamber $\mathfrak{C}_a$ is a cone in Θ such that the admissible G-iraffes are θ-stable for $\theta \in \mathfrak{C}_a$. By Section 5.1 and considering the torus invariant G-constellations which x acts trivially on, it is immediate that the chamber structure in Θ is finer than the Weyl chamber structure of A_{r-1}. Therefore the admissible Weyl chamber is a full chamber in the stability parameter space Θ (see Section 5.2).

We have proved the following proposition:

Proposition 4.2.14. *For the set $\mathfrak{G}$ of G-iraffes in Corollary 4.1.8, there exists an open Weyl chamber $\mathfrak{C}_a \subset \Theta$ such that Γ is θ-stable if $\Gamma \in \mathfrak{G}$ and $\theta \in \mathfrak{C}_a$. Furthermore, the chamber $\mathfrak{C}_a$ is a full chamber in Θ.*

From Step 3, we make the following conjecture:

Conjecture 4.2.15. *The chamber structure of the GIT stability parameter space Θ of G-constellations coincides with the Weyl chamber structure of A_{r-1}.*

4.3 Main theorem

Theorem 4.3.1 (Main Theorem)**.** *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. Let $\Sigma_{\max}$ be the set of 3-dimensional cones in the fan of the economic resolution Y of $X = \mathbb{C}^3/G$. Then there exist a set $\mathfrak{G}$ of G-iraffes and $\theta \in \Theta$ such that:*

(i) there exists a bijective map $\Sigma_{\max} \to \mathfrak{G}$ sending σ to Γ_σ with $S(\Gamma_\sigma) = \sigma^\vee \cap M$.

(ii) every Γ_σ is θ-stable if $\Gamma_\sigma \in \mathfrak{G}$.

Thus Y is isomorphic to $\bigcup_{\Gamma \in \mathfrak{G}} U(\Gamma)$. In particular, $U(\Gamma)$ is smooth for any $\Gamma \in \mathfrak{G}$.

Proof. Corollary 4.1.8 shows that there exists a set $\mathfrak{G}$ of G-iraffes satisfying the condition (i). For the set $\mathfrak{G}$, Lemma 4.2.4 shows that there exists a stability parameter θ satisfying the condition (ii). $\square$

Corollary 4.3.2. *With the notation as Theorem 4.3.1, the economic resolution Y is isomorphic to the birational component Y_θ of the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations.*

Proof. The main theorem proves that the economic resolution Y is isomorphic to $\bigcup_{\Gamma\in\mathfrak{G}} U(\Gamma)$. From Proposition 2.5.7, there exists an open immersion from Y to Y_θ. This open immersion is a closed embedding because both Y and Y_θ are projective over X. Since both Y and Y_θ are 3-dimensional and irreducible, this embedding is an isomorphism.

$$\begin{tikzcd} Y \cong \bigcup_{\Gamma\in\mathfrak{G}} U(\Gamma) \arrow[r, hook] \arrow[dr, "\text{proper}"'] & Y_\theta \arrow[d, "\text{proj.}"] \\ & X \end{tikzcd}$$

$\square$

By the construction of this family, we have seen that elements in Γ form a $\mathbb{C}$-basis of the G-constellation over $p \in U(\Gamma)$.

Conjecture 4.3.3. *The moduli space $\mathcal{M}_\theta$ is irreducible. In particular, any θ-stable G-graph Γ is in the set $\mathfrak{G}$ in Theorem 4.3.1.*

If this conjecture holds, then the moduli space $\mathcal{M}_\theta$ is isomorphic to the economic resolution. From Remark 4.1.9, it is enough to show that torus invariant θ-stable G-constellations corresponding to $\Gamma \in \mathfrak{G}$ are all of θ-stable torus invariant G-constellation. In the case $G = \frac{1}{2k+1}(1,2,2k-1)$, we can prove that Conjecture 4.3.3 is true so $\mathcal{M}_\theta$ is isomorphic to the economic resolution for $\theta \in \mathfrak{C}_a$. We hope to establish this more generally in future work.

Remark 4.3.4 (Link to [16])**.** Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(1,a,r-a)$ and $A \subset \mathrm{SL}_2(\mathbb{C})$ the group of type $\frac{1}{r}(a,r-a)$.

Kędzierski [16] describes a Weyl chamber $\mathfrak{C} \subset \Theta$ such that the normalization of Y_θ is isomorphic to the economic resolution Y of $X = \mathbb{C}^3/G$ for $\theta \in \mathfrak{C}$. In his description, he did not use the root system A_{r-1}, but a set of inequalities, however his description is essentially the same as using the root system.

His tactic in [16] is using the anticanonical system described in Section 3.4 (See also Section 5.2):

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ D \arrow[r, hook] & X. \end{tikzcd}$$

More precisely, the elephant S given by $x = 0$ in Y is the minimal resolution of D, where D is the divisor given by $x = 0$ in $X = \mathbb{C}^3/G$. By the 2-dimensional McKay correspondence, S is isomorphic to the moduli space of θ-stable A-constellations and there is a universal family

$$\mathcal{U} = \bigoplus_{i \in I} \mathcal{U}_i$$

over S. He constructed line bundles $\mathcal{L}_i$ on Y such that $\mathcal{L}_i|_S \cong \mathcal{U}_i$ for each $i \in I$. He proved that the collection of the line bundles is a gnat family [20] and that the family induces a bijective morphism from Y to Y_θ. ♦

4.4 Universal families

In the previous sections, we assigned a θ-stable G-graph Γ_σ to each full dimensional cone σ of the fan of the economic resolution Y of $X = \mathbb{C}^3/G$, where G is of type $\frac{1}{r}(1, a, r-a)$ with a coprime to r. This section describes the *universal family* over the economic resolution Y.

Let ρ be an irreducible representation of G. From the data (σ, Γ_σ), for each full dimensional cone σ, there exists a unique Laurent monomial $\mathbf{m}_\sigma \in \Gamma_\sigma$ whose weight is ρ. The data $\{\mathbf{m}_\sigma\}$ is called the *canonical data* of ρ.

Remark 4.4.1. This canonical data gives a line bundle, which is called a universal family over $Y_\theta = G\text{-Hilb}\,\mathbb{C}^3$ if $a = 1$ or $r - 1$. ♦

Proposition 4.4.2. *Let ρ be a fixed irreducible representation of G. The canonical data $\{\mathbf{m}_\sigma\}$ of ρ gives a line bundle $\mathcal{L}_\rho$ on Y satisfying $\mathcal{L}_\rho|_{U_\sigma} \cong \mathcal{O}_{U_\sigma}(\operatorname{div} \mathbf{m}_\rho^{-1})$. In other words, $\mathcal{L}_\rho$ is the line bundle corresponding to the Cartier divisor D_ρ defined by $D_\rho|_{U_\sigma} = \operatorname{div} \mathbf{m}_\rho^{-1}|_{U_\sigma}$ for all σ.*

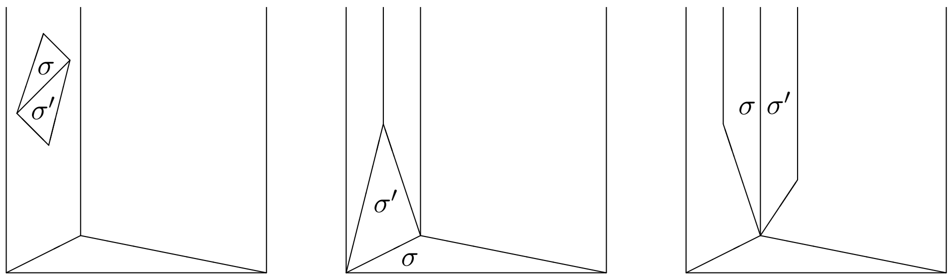
Proof. From general toric geometry (see e.g. [3]), it suffices to show that $\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}}$ vanishes on the intersection $\sigma \cap \sigma'$ for any two adjacent cones σ, σ'. Suppose that the intersection is the cone generated by $\mathbf{u}_1, \mathbf{u}_2 \in L$ and then it should be shown that $\langle \mathbf{u}_i, \frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}} \rangle$ is zero for $i = 1, 2$. Set $\mathbf{m}_\sigma = x^{m_1} y^{m_2} z^{m_3}$ and $\mathbf{m}_{\sigma'} = x^{m'_1} y^{m'_2} z^{m'_3}$. There are four cases:

(1) Both σ and σ' are cones in either the left side or the right side.

(2) One of them is the cone on the central side and the other is the cone on the central side of the left side.

(3) One of them is the cone on the central side and the other is the cone on the central side on the right side.

(4) One of them is the most right cone of the left side and the other is the most left cone of the right side.



Case (1) Case (2) Case (4)

Figure 4.4.1: Four cases for two full dimensional cones in the fan of Y

Case (1) Assume that the cones are on the left side. Let ϕ_2 be the left round down function of the weighted blowup with weight $(1, a, r-a)$. Since the weights of $\mathbf{m}_\sigma$ and $\mathbf{m}_\sigma$ are equal to ρ, $\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}} = \phi_2(\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}})$. By induction on r, it follows that $\langle \mathbf{u}_i, \frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}} \rangle = 0$.

Case (2) Assume that σ is the cone on the central side and that σ' is the cone on the central side of the left side. Note that the G-graph for σ is $\{1, x, x^2, \ldots, x^{r-1}\}$ and that the G-graph for σ' is

$$\left\{\mathbf{m} \in \overline{M} \,\middle|\, \phi_2(\mathbf{m}) \in \{1, \xi, \ldots, \xi^{a-1}\}\right\}.$$

Thus, with the fact that both $\mathbf{m}_\sigma$ and $\mathbf{m}_{\sigma'}$ have the same weights,

$$\begin{aligned}
\mathbf{m}_\sigma &= x^{m_1} && \text{for some } 0 \le m_1 < r, \\
\mathbf{m}_{\sigma'} &= x^{m'_1} y^{m'_2} && \text{for some } 0 \le m'_1 < a \quad \text{with } m'_1 + a m'_2 = m_1.
\end{aligned}$$

Since $\sigma \cap \sigma' = \text{Cone}\left((0,0,1), \frac{1}{r}(1, a, r-a)\right)$, the Laurent monomial $\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}}$ vanishes on the intersection.

Case (3) Case (3) is similar to Case (2).

Case (4) Assume that σ is the most right cone in the left side and that σ' is the most left cone in the right side. Note that $\sigma \cap \sigma'$ is the cone generated by $(1,0,0), \frac{1}{r}(1,a,r-a)$. Similarly to Case (2), note that

$$\mathbf{m}_\sigma = y^{m_2} z^{m_3}, \quad \mathbf{m}_{\sigma'} = y^{m'_2} z^{m'_3}$$

with $am_2 + (r-a)m_3 = am'_2 + (r-a)m'_3$. Hence it follows that $\frac{\mathbf{m}_\sigma}{\mathbf{m}_{\sigma'}}$ vanishes on the intersection. $\square$

Remark 4.4.3. For the trivial representation ρ_0, $\mathbf{1}$ is in every G-graph and hence the line bundle for the trivial representation is $\mathcal{O}_Y$. The direct sum of all such line bundles

$$\mathcal{L} = \bigoplus_{\rho \in G^\vee} \mathcal{L}_\rho$$

is a *gnat family* in the sense of [20], which is the same family in [16]. ♦

Example 4.4.4. Let G be the group of type $\frac{1}{7}(1,3,4)$ as in Example 2.4.3. Let ρ be the irreducible representation of G with weight 1. Consider the line bundle $\mathcal{L}_\rho$ as in Proposition 4.4.2. In Figure 4.4.2, the monomial in a maximal cone σ is a unique element in Γ_σ whose weight is 1. ♦

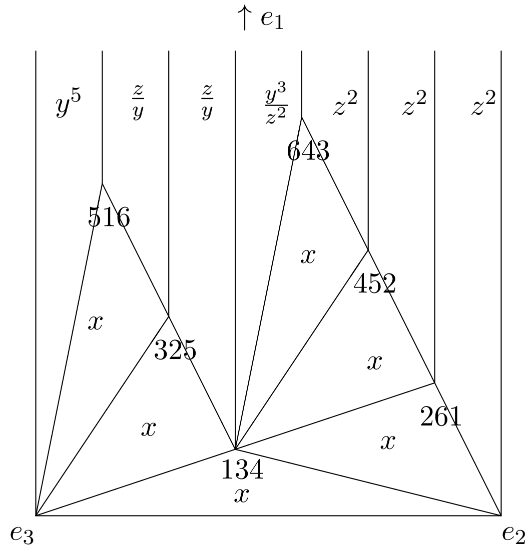


Figure 4.4.2: Elements of weight 1 in Γ_σ for $\frac{1}{7}(1,3,4)$

4.5 Example: type $\frac{1}{12}(1,7,5)$

In this section, as a concrete example, we calculate the set of G-iraffes and the admissible set of simple roots Δ_a for the group G of type $\frac{1}{12}(1,7,5)$.

Let G be the finite group of type $\frac{1}{12}(1,7,5)$ with eigencoordinates x, y, z and L the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{12}(1,7,5)$. Let X denote the quotient variety $\mathbb{C}^3/G$ and Y the economic resolution of X. The toric fan Σ of Y is shown in Figure 4.5.1.

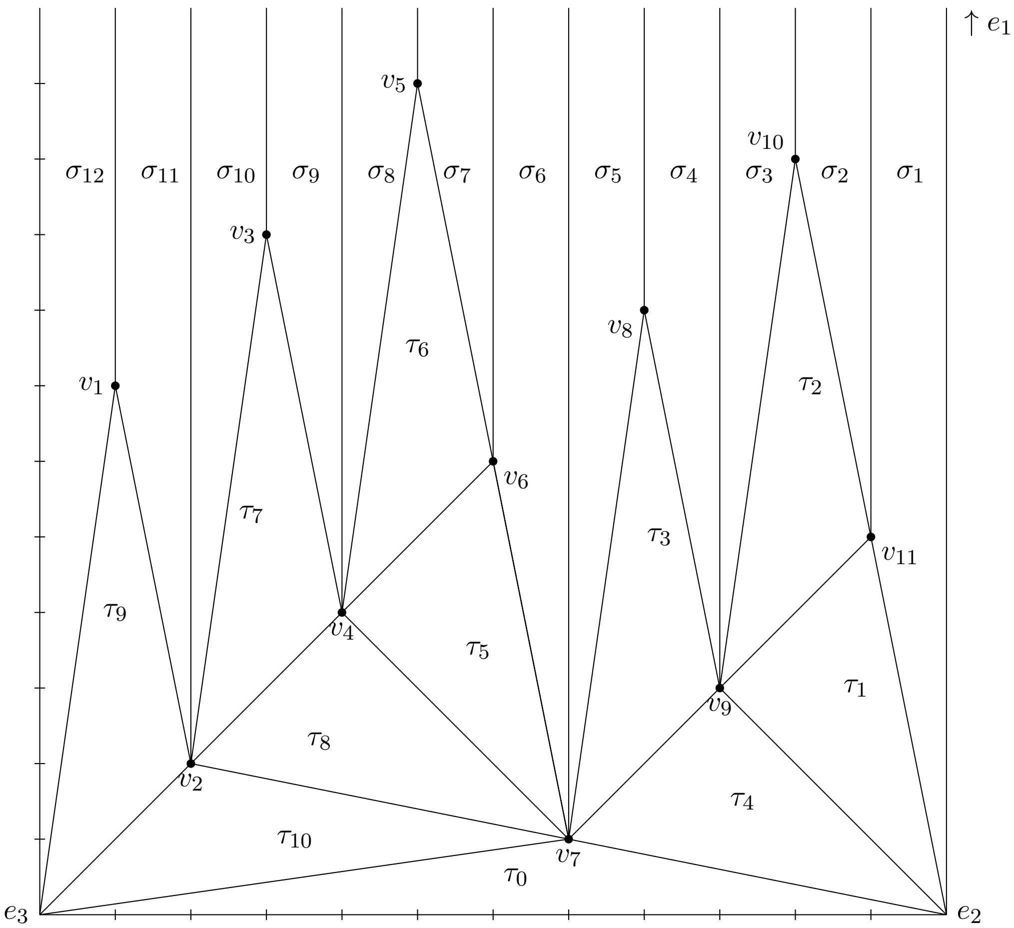


Figure 4.5.1: Toric fan of the economic resolution for $\frac{1}{12}(1,7,5)$

To use the recursion process as in Section 4.1, we need to investigate the cases of type $\frac{1}{7}(1,2,5)$ and of type $\frac{1}{5}(1,2,3)$. Let G_2 be the group of type $\frac{1}{7}(1,2,5)$ with eigencoordinates ξ_2, η_2, ζ_2 and G_3 be the group of type $\frac{1}{5}(1,2,3)$ with eigencoordinates ξ_3, η_3, ζ_3. Consider the toric fans Σ_2 and Σ_3 of the economic resolutions for the type $\frac{1}{7}(1,2,5)$ and the type $\frac{1}{5}(1,2,3)$, respectively.

4.5.1 G-iraffes

We now calculate G-iraffes corresponding to two full dimensional cones in Σ:

$$\begin{aligned}\sigma_4 &= \mathrm{Cone}\ \left(\tfrac{1}{12}(12,0,0), \tfrac{1}{12}(3,9,3), \tfrac{1}{12}(8,8,4)\right),\\ \tau_3 &= \mathrm{Cone}\left(\tfrac{1}{12}(1,7,5), \tfrac{1}{12}(3,9,3), \tfrac{1}{12}(8,8,4)\right).\end{aligned}$$

Note that the cones σ_4, τ_3 are on the right side of the lowest vector $v = \frac{1}{12}(1,7,5)$. Their corresponding cones σ_4', τ_3' in Σ_3 to σ_4, τ_3, respectively are

$$\sigma_4' = \mathrm{Cone}\left(\tfrac{1}{5}(5,0,0), \tfrac{1}{5}(1,2,3), \tfrac{1}{5}(1,1,4)\right), \tag{4.5.1}$$

$$\tau_3' = \mathrm{Cone}\left(\tfrac{1}{5}(0,0,5), \tfrac{1}{5}(1,2,3), \tfrac{1}{5}(1,1,4)\right). \tag{4.5.2}$$

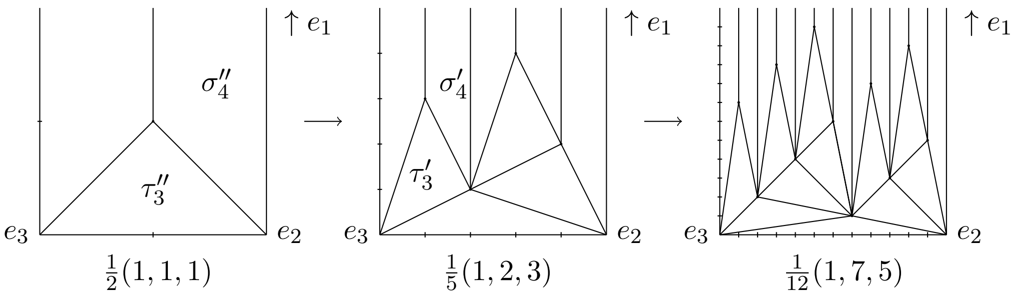


Figure 4.5.2: Recursion process for $\frac{1}{12}(1,7,5)$

Observe that the cones σ_4', τ_3' are on the left side of Σ_3. To use the recursion, let G_{32} be the group of type $\frac{1}{2}(1,1,1)$ with eigencoordinates $\xi_{32}, \eta_{32}, \zeta_{32}$. Let Σ_{32} denote the fan of the economic resolution of the quotient $\mathbb{C}^2/G_{32}$. In Σ_{32}, there exist two cones σ_4'', τ_3'' corresponding to σ_4', τ_3', respectively:

$$\begin{aligned}\sigma_4'' &= \mathrm{Cone}\left(\tfrac{1}{2}(2,0,0), \tfrac{1}{2}(0,2,0), \tfrac{1}{2}(1,1,1)\right),\\ \tau_3'' &= \mathrm{Cone}\left(\tfrac{1}{2}(0,0,2), \tfrac{1}{2}(0,2,0), \tfrac{1}{2}(1,1,1)\right).\end{aligned}$$

As is in Example 4.1.2, the G_{32}-graphs Γ_4'' and Γ_3'' corresponding to σ_4'', τ_3''

are

$$\begin{aligned}\Gamma_4'' &= \Big\{1, \zeta_{23}\Big\},\\ \Gamma_3'' &= \Big\{1, \xi_{23}\Big\}.\end{aligned}$$

Using the left round down function ϕ_{32} for $\frac{1}{5}(1,2,3)$

$$\phi_{32}\colon \xi_3^a\eta_3^b\zeta_3^c \;\mapsto\; \xi_{32}^a\eta_{32}^{\lfloor\frac{a+2b+3c}{5}\rfloor}\zeta_{32}^c,$$

we can see that the corresponding G_3-graphs Γ_4' and Γ_3' corresponding to σ_4', τ_3' are

$$\begin{aligned}\Gamma_4' &\stackrel{\text{def}}{=} \phi_{32}^{-1}(\Gamma_4'') &= \Big\{1,\, \eta_3,\, \eta_3^2,\, \zeta_3,\, \tfrac{\zeta_3}{\eta_3}\Big\},\\ \Gamma_3' &\stackrel{\text{def}}{=} \phi_{32}^{-1}(\Gamma_2'') &= \Big\{1,\, \eta_3,\, \eta_3^2,\, \xi_3,\, \xi_3\eta_3\Big\}.\end{aligned}$$

To get the corresponding G-iraffes Γ_4, Γ_3 to σ_4, τ_3, respectively, we use the right round function ϕ_3 for $\frac{1}{12}(1,7,5)$:

$$\phi_3\colon x^ay^bz^c \;\mapsto\; \xi_3^a\eta_3^b\zeta_3^{\lfloor\frac{a+7b+5c}{12}\rfloor}.$$

We get

$$\begin{aligned}\Gamma_4 &\stackrel{\text{def}}{=} \phi_3^{-1}(\Gamma_4') &= \Big\{1, y, \tfrac{y}{z}, \tfrac{y^2}{z}, \tfrac{y^2}{z^2}, z, z^2, z^3, z^4, \tfrac{z^4}{y}, \tfrac{z^5}{y}, \tfrac{z^6}{y}\Big\},\\ \Gamma_3 &\stackrel{\text{def}}{=} \phi_3^{-1}(\Gamma_2') &= \Big\{1, x, xz, xz^2, xy, \tfrac{xy}{z}, y, \tfrac{y}{z}, \tfrac{y^2}{z}, \tfrac{y^2}{z^2}, z, z^2\Big\}.\end{aligned}$$

Let us consider the following two cones:

$$\begin{aligned}\sigma_9 &= \mathrm{Cone}\left(\tfrac{1}{12}(12,0,0), \tfrac{1}{12}(9,3,9), \tfrac{1}{12}(4,4,8)\right),\\ \tau_7 &= \mathrm{Cone}\left(\tfrac{1}{12}(2,2,10), \tfrac{1}{12}(9,3,9), \tfrac{1}{12}(4,4,8)\right).\end{aligned}$$

Observe that the cones σ_9, τ_7 are on the left side of v. The corresponding cones σ_9', τ_7' in Σ_2 to σ_9, τ_7, respectively are

$$\begin{aligned}\sigma_9' &= \mathrm{Cone}\left(\tfrac{1}{7}(12,0,0), \tfrac{1}{7}(5,3,4), \tfrac{1}{7}(2,4,3)\right),\\ \tau_7' &= \mathrm{Cone}\left(\tfrac{1}{7}(1,2,5), \tfrac{1}{7}(5,3,4), \tfrac{1}{7}(2,4,3)\right).\end{aligned}$$

Note that the cones σ_9', τ_7' are on the right side of the fan Σ and that the right side is equal to the fan Σ_3 of the economic resolution for $\frac{1}{5}(1,2,3)$. Moreover, the cones in Σ_3 corresponding to σ_9', τ_7' are σ_4', τ_2', respectively in (4.5.1). Thus we have the corresponding G_{23}-graphs Γ''_9, Γ''_7 are:

$$\Gamma''_9 = \Big\{1,\ \eta_{23},\ \eta_{23}^2,\ \zeta_{23},\ \tfrac{\zeta_{23}}{\eta_{23}}\Big\},$$
$$\Gamma''_7 = \Big\{1,\ \xi_{23},\ \xi_{23}\eta_{23},\ \eta_{23},\ \eta_{23}^2\Big\},$$

where G_{23} is the group of type $\frac{1}{5}(1,2,3)$ with eigencoordinates $\xi_{23}, \eta_{23}, \zeta_{23}$. Using the right round down function ϕ_{23} for $\frac{1}{7}(1,2,5)$

$$\phi_{23}\colon \xi_2^a\eta_2^b\zeta_2^c \ \mapsto\ \xi_{23}^a\eta_{23}^b\zeta_{23}^{\lfloor\frac{a+2b+5c}{7}\rfloor},$$

we can calculate the corresponding G_2-graphs to σ'_9, τ'_7:

$$\begin{aligned}
\Gamma'_9 &\overset{\text{def}}{=} \phi_{23}^{-1}(\Gamma''_9) &&= \Big\{1,\ \eta_2,\ \eta_2^2,\ \zeta_2,\ \zeta_2^2,\ \tfrac{\zeta_2^2}{\eta_2},\ \tfrac{\zeta_2^3}{\eta_2}\Big\},\\
\Gamma'_7 &\overset{\text{def}}{=} \phi_{23}^{-1}(\Gamma''_7) &&= \Big\{1,\ \xi_2,\ \xi_2\eta_2,\ \xi_2\zeta_2,\ \eta_2,\ \eta_2^2,\ \zeta_2,\ \zeta_2^2\Big\}.
\end{aligned}$$

Lastly, from the left round down function ϕ_2 for $\frac{1}{12}(1,7,5)$

$$\phi_2\colon x^ay^bz^c \ \mapsto\ \xi_2^a\eta_2^{\lfloor\frac{a+7b+5c}{12}\rfloor}\zeta_2^c,$$

it follows that the G-iraffes Γ_9, Γ_7 corresponding to σ_9, τ_7 are:

$$\Gamma_9 = \Big\{1, y, y^2, y^3, y^4, y^5, z, z^2, \tfrac{z^2}{y}, \tfrac{z^2}{y^2}, \tfrac{z^2}{y^3}, \tfrac{z^3}{y^3}\Big\},$$
$$\Gamma_7 = \Big\{1, x, xy, xy^2, xy^3, xz, y, y^2, y^3, y^4, y^5, z\Big\}.$$

For each $0 \le i \le 12$, let v_i denote the lattice point $\frac{1}{12}(\overline{7i}, i, 12-i)$ in L. For the cones σ in Figure 4.5.1 in page 61, Table 4.5.1 in page 66 shows the corresponding G-iraffe Γ_σ.

4.5.2 Admissible set of simple roots

Now we calculate the admissible set of simple roots for $\frac{1}{12}(1,7,5)$. Since for the group of type $\frac{1}{r}(1,r-1,1)$, the economic resolution is G-Hilb, note that the admissible sets of simple roots for $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,2,1)$ are $\{\varepsilon_1-\varepsilon_0\}, \{\varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_0\}$, respectively. By the identification (4.2.7), that the admissible sets of simple roots for $\frac{1}{5}(1,2,3)$ is

$$\{\varepsilon_3-\varepsilon_4,\ \underline{\varepsilon_4-\varepsilon_1},\ \varepsilon_1-\varepsilon_2,\ \varepsilon_2-\varepsilon_0\},$$

where the underlined root is the added root as in (4.2.8). Similarly, the admissible sets of simple roots for $\frac{1}{7}(1,2,5)$ is

$$\{\varepsilon_5 - \varepsilon_6,\ \underline{\varepsilon_6 - \varepsilon_3},\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_4 - \varepsilon_1,\ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_0\}.$$

Lastly, the admissible set of simple roots for $\frac{1}{12}(1,7,5)$ is

$$\left\{\begin{matrix} \varepsilon_5 - \varepsilon_6,\ \varepsilon_6 - \varepsilon_{10},\ \varepsilon_{10} - \varepsilon_{11},\ \varepsilon_{11} - \varepsilon_8,\ \varepsilon_8 - \varepsilon_9,\ \varepsilon_9 - \varepsilon_7, \\ \underline{\varepsilon_7 - \varepsilon_3},\ \varepsilon_3 - \varepsilon_4,\ \varepsilon_4 - \varepsilon_1,\ \varepsilon_1 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_0 \end{matrix}\right\}.$$

Note that corresponding permutation ω is

$$\omega = \begin{pmatrix} 0 & 7 & 2 & 9 & 4 & 11 & 6 & 1 & 8 & 3 & 10 & 5 \\ 0 & 2 & 1 & 4 & 3 & 7 & 9 & 8 & 11 & 10 & 6 & 5 \end{pmatrix}.$$

With the dual basis $\{\theta_i\}$ with respect to $\{\alpha_i\}$, the row vectors of the following matrix is the rays of the admissible Weyl chamber $\mathfrak{C}_a$:

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
-1 & -1 & -1 & 0 & -1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
-1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
-1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
-1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
-1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Cone	Generators	G-iraffes Γ_σ	Coordinates on U_σ
σ_1	e_1, e_2, v_{11}	$1, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9, z^{10}, z^{11}$	$\frac{x}{z^5}, \frac{y}{z^{11}}, z^{12}$
σ_2	e_1, v_{10}, v_{11}	$1, y, \frac{y}{z}, z, z^2, z^3, z^4, z^5, z^6, z^7, z^8, z^9$	$\frac{x}{z^5}, \frac{y^2}{z^{10}}, \frac{z^{11}}{y}$
σ_3	e_1, v_9, v_{10}	$1, y, \frac{y}{z}, \frac{y^2}{z}, \frac{y^2}{z^2}, \frac{y^2}{z^3}, \frac{y^2}{z^4}, \frac{y^2}{z^5}, z, z^2, z^3, z^4$	$\frac{xz^5}{y^2}, \frac{y^3}{z^9}, \frac{z^{10}}{y^2}$
σ_4	e_1, v_8, v_9	$1, y, \frac{y}{z}, \frac{y^2}{z}, \frac{y^2}{z^2}, z, z^2, z^3, z^4, \frac{z^4}{y}, \frac{z^5}{y}, \frac{z^6}{y}$	$\frac{xy}{z^4}, \frac{y^4}{z^8}, \frac{z^9}{y^3}$
σ_5	e_1, v_7, v_8	$1, y, \frac{y}{z}, \frac{y^2}{z}, \frac{y^2}{z^2}, \frac{y^3}{z^2}, \frac{y^3}{z^3}, \frac{y^3}{z^4}, \frac{y^4}{z^4}, \frac{y^4}{z^5}, z, z^2$	$\frac{xz^4}{y^3}, \frac{y^5}{z^7}, \frac{z^8}{y^4}$
σ_6	e_1, v_6, v_7	$1, y, z, z^2, \frac{z^2}{y}, \frac{z^3}{y}, \frac{z^3}{y^2}, \frac{z^4}{y^2}, \frac{z^5}{y^2}, \frac{z^5}{y^3}, \frac{z^6}{y^3}, \frac{z^6}{y^4}$	$\frac{xy^2}{z^3}, \frac{y^6}{z^6}, \frac{z^7}{y^5}$
σ_7	e_1, v_5, v_6	$1, y, y^2, y^3, z, z^2, \frac{z^2}{y}, \frac{z^3}{y}, \frac{z^3}{y^2}, \frac{z^4}{y^2}, \frac{z^5}{y^2}, \frac{z^5}{y^3}$	$\frac{xy^2}{z^3}, \frac{y^7}{z^5}, \frac{z^6}{y^6}$
σ_8	e_1, v_4, v_5	$1, y, y^2, y^3, y^4, y^5, \frac{y^5}{z}, \frac{y^5}{z^2}, \frac{y^6}{z^2}, z, z^2, \frac{z^2}{y}$	$\frac{xz^2}{y^5}, \frac{y^8}{z^4}, \frac{z^5}{y^7}$
σ_9	e_1, v_3, v_4	$1, y, y^2, y^3, y^4, y^5, z, z^2, \frac{z^2}{y}, \frac{z^2}{y^2}, \frac{z^2}{y^3}, \frac{z^3}{y^3}$	$\frac{xy^3}{z^2}, \frac{y^9}{z^3}, \frac{z^4}{y^8}$
σ_{10}	e_1, v_2, v_3	$1, y, y^2, y^3, y^4, y^5, y^6, \frac{y^6}{z}, \frac{y^7}{z}, \frac{y^8}{z}, \frac{y^9}{z}, z$	$\frac{xz}{y^6}, \frac{y^{10}}{z^2}, \frac{z^3}{y^9}$
σ_{11}	e_1, v_1, v_2	$1, y, y^2, y^3, y^4, y^5, y^6, z, \frac{z}{y}, \frac{z}{y^2}, \frac{z}{y^3}, \frac{z}{y^4}$	$\frac{xy^4}{z}, \frac{y^{11}}{z^1}, \frac{z^2}{y^{10}}$
σ_{12}	e_1, e_3, v_1	$1, y, y^2, y^3, y^4, y^5, y^6, y^7, y^8, y^9, y^{10}, y^{11}$	$\frac{x}{y^7}, y^{12}, \frac{z}{y^{11}}$
τ_1	e_2, v_9, v_{11}	$1, x, xz, xz^2, xz^3, xz^4, x^2, x^2z, z, z^2, z^3, z^4$	$\frac{x^3}{z^3}, \frac{y}{x^2z}, \frac{z^5}{x}$
τ_2	v_9, v_{10}, v_{11}	$1, x, z, xz, z^2, xz^2, z^3, xz^3, z^4, xz^4, y, \frac{y}{z}$	$\frac{x^2z}{y}, \frac{y^2}{xz^5}, \frac{z^5}{x}$
τ_3	v_7, v_8, v_9	$1, x, xy, \frac{xy}{z}, xz, xz^2, y, \frac{y}{z}, \frac{y^2}{z}, \frac{y^2}{z^2}, z, z^2$	$\frac{x^2z}{y}, \frac{y^3}{xz^4}, \frac{z^4}{xy}$
τ_4	e_2, v_7, v_9	$1, x, x^2, x^3, x^4, xz, xz^2, x^2z, x^3z, x^4z, z, z^2$	$\frac{x^5}{z}, \frac{y}{x^2z}, \frac{z^3}{x^3}$
τ_5	v_4, v_6, v_7	$1, x, xy, xz, xz^2, \frac{xz^2}{y}, x^2, x^2y, y, z, z^2, \frac{z^2}{y}$	$\frac{x^3y}{z^2}, \frac{y^2}{x^2}, \frac{z^3}{xy}$
τ_6	v_4, v_5, v_6	$1, x, xy, xz, xz^2, \frac{xz^2}{y}, y, y^2, y^3, z, z^2, \frac{z^2}{y}$	$\frac{x^2}{y^2}, \frac{y^5}{xz^2}, \frac{z^3}{xy^2}$
τ_7	v_2, v_3, v_4	$1, x, xy, xy^2, xy^3, xz, y, y^2, y^3, y^4, y^5, z$	$\frac{x^2}{y^2}, \frac{y^6}{xz}, \frac{z^2}{xy^3}$
τ_8	v_2, v_4, v_7	$1, x, xy, xz, x^2, x^2y, x^3, x^3y, x^4, x^4y, y, z$	$\frac{x^5}{z}, \frac{y^2}{x^2}, \frac{z^2}{x^3y}$
τ_9	e_3, v_1, v_2	$1, x, xy, xy^2, xy^3, xy^4, y, y^2, y^3, y^4, y^5, y^6$	$\frac{x^2}{y^2}, \frac{y^7}{x}, \frac{z}{xy^4}$
τ_{10}	e_3, v_2, v_7	$1, x, xy, x^2, x^2y, x^3, x^3y, x^4, x^4y, x^5, x^6, y$	$\frac{x^7}{y}, \frac{y^2}{x^2}, \frac{z}{x^5}$
τ_0	e_2, e_3, v_7	$1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, x^{10}, x^{11}$	$x^{12}, \frac{y}{x^7}, \frac{z}{x^5}$

Table 4.5.1: G-iraffes for $G = \frac{1}{12}(1,7,5)$

Chapter 5

Further discussion

5.1 Torus invariant G-constellations for type $\frac{1}{r}(1,-1)$

For a finite subgroup of $G \subset \mathrm{SL}_2(\mathbb{C})$, it is well known that the GIT stability parameter space of G-constellations has the same chamber structure as the Weyl chamber structure for the root system corresponding to the type of the group G (see [6,19,30]). In this section, we describe explicitly torus invariant G-constellations for each chamber of the GIT stability parameter space where the group G is of type $\frac{1}{r}(1,-1)$.

5.1.1 Chambers of GIT stability parameter spaces

Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be the finite group of type $\frac{1}{r}(1,-1)$ with coordinates y,z. Set $\mathbf{T} = (\mathbb{C}^\times)^2$. Let ρ_i be the irreducible representation of G whose weight is i. We can identify $I := \mathrm{Irr}(G)$ with $\mathbb{Z}/r\mathbb{Z}$.

Let $\left\{\varepsilon_i \,\middle|\, i \in I\right\}$ be an orthonormal basis of $\mathbb{Q}^r$, i.e. $\langle \varepsilon_i, \varepsilon_j\rangle = \delta_{ij}$. Define

$$\Phi := \left\{\varepsilon_i - \varepsilon_j \,\middle|\, i,j \in I, i \neq j\right\}.$$

Let $\mathfrak{h}^*$ be the subspace of $\mathbb{Q}^r$ generated Φ. Elements in Φ are called *roots*. For each nonzero $i \in I$, set $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$. Set $S_r := \left\{\omega \,\middle|\, \omega \text{ is a permutation of } I\right\}$.

The stability parameter space Θ can be identified with the dual space of $\mathfrak{h}^*$ by considering roots as dimension vectors. Note that $\Delta_+ = \left\{\alpha_i \,\middle|\, i \in I, i \neq 0\right\}$ is a set of simple roots and the corresponding Weyl chamber $\mathfrak{C}_+$ is

$$\begin{aligned}\mathfrak{C}_+ &= \left\{\theta \in \Theta \,\middle|\, \theta(\alpha) > 0 \quad \forall \alpha \in \Delta_+\right\}\\ &= \left\{\theta \in \Theta \,\middle|\, \theta(\rho_i) > 0 \quad \forall \rho_i \neq \rho_0\right\},\end{aligned}$$

which is the chamber Θ_+ for G-$\mathrm{Hilb}\,\mathbb{C}^2$ in (2.2.10). Let $\{\theta_i\}_{i=1}^{r-1}$ be the dual basis of the GIT parameter space Θ with respect to $\{\alpha_i\}_{i=1}^{r-1}$, i.e. $\theta_i(\alpha_j) = \delta_{ij}$. Using the basis $\{\varepsilon_i\}$ with the standard inner product, we can write:

$$\theta_i = -\sum_{j=0}^{i-1} \varepsilon_j$$

for $1 \leq i \leq r-1$. Set $\theta_0 = -\sum_{i=1}^{r-1} \theta_i$. As is standard, we can present the rays of the Weyl chamber $\mathfrak{C}(\omega)$ using this basis and the permutation ω: the rays are generated by the following vectors

$$\sum_{j=0}^{i-1} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right) \tag{5.1.1}$$

for $i = 1, 2, \ldots, r-1$, which is the dual basis with respect to the set of simple roots $\Delta(\omega)$.

5.1.2 Lacings for each chamber

On the other hand, any torus invariant (connected) G-constellation is given by a *lacing*. The following definition originates from the idea in calculations due to Nolla [26] and Reid.

Definition 5.1.2. A *lacing* Λ for $G = \frac{1}{r}(1, -1)$ consists of two subsets (Λ^y, Λ^z) of $\mathrm{Irr}(G) \cong \mathbb{Z}/r\mathbb{Z}$ such that:

(i) $|\Lambda^y| + |\Lambda^z| = r + 1$ where $|\cdot|$ is the cardinality of the set.

(ii) if $i \notin \Lambda^y$ for $i \in I$, then $i + 1 \in \Lambda^z$.

For a generic $\theta \in \Theta$, a lacing Λ is said to be *θ-stable* if the G-constellation corresponding to Λ is θ-stable.

Proposition 5.1.3. *Let G be the finite group of type $\frac{1}{r}(1, -1)$. Let θ be a generic parameter in Θ. There exists a 1-to-1 correspondence between the set of isomorphism classes of θ-stable torus invariant G-constellations and the set of θ-stable lacings.*

Proof. Let $\mathcal{F}$ be a θ-stable torus invariant G-constellation. Define $\Lambda = (\Lambda^y, \Lambda^z)$ to be

$$\begin{aligned} \Lambda^y &:= \left\{ i \in I \,\middle|\, y_i = 0 \right\}, \\ \Lambda^z &:= \left\{ i \in I \,\middle|\, z_i = 0 \right\}, \end{aligned}$$

where y_i (resp. z_i) is the y-action (resp. z-action) on the basis of $\mathbb{C}e_{\rho_i}$. Then Λ is a lacing. Indeed, as the monomial yz is G-invariant, it gives a cycle around each vertex, so $y_i z_{i+1}$ must be zero by Lemma 2.6.4, i.e. one of y_i and z_{i+1} must be zero.

Thus the condition (ii) in Definition 5.1.2 is satisfied. For the condition (i), note that as $\mathcal{F}$ is θ-stable, it is connected, and so we need at least $r-1$ nonzero arrows. This shows that Λ is a lacing.

For the converse, let $\Lambda = (\Lambda^y, \Lambda^z)$ be a lacing. We define a G-constellation $\mathcal{F}$ as follows: the G-constellation $\mathcal{F}$ is $\oplus_{i\in I}\mathbb{C}e_i$ as a $\mathbb{C}$-vector space where e_i is a basis of $\mathbb{C}\rho_i$, whose $\mathbb{C}[y,z]$-module structure is given by:

$$\begin{aligned} y * e_i &= \begin{cases} e_{i+1} & \text{if } i \notin \Lambda^y, \\ 0 & \text{if } i \in \Lambda^y, \end{cases} \\ z * e_i &= \begin{cases} e_{i-1} & \text{if } i \notin \Lambda^z, \\ 0 & \text{if } i \in \Lambda^z. \end{cases} \end{aligned}$$

One can easily show that $\mathcal{F}$ is a torus invariant G-constellation. $\square$

Remark 5.1.4. For those familiar with McKay quivers, the set Λ^y is the index set for vanishing y_i and the set Λ^z is the index set for vanishing z_i. The corresponding G-constellation does not have any (undirected) cycle so it is a torus invariant G-constellation by Lemma 2.6.4. ♦

Lemma 5.1.5. *Let $\Lambda = (\Lambda^y, \Lambda^z)$ be a lacing and $\theta \in \Theta$ generic. Then Λ is θ-stable if and only if $\theta(\varepsilon_k - \varepsilon_{l-1}) \geq 0$, for any $k \in \Lambda^y$ and $l \in \Lambda^z$.*

This Lemma can be proved by the same method as the proof of Lemma 8.3. in [16].

Proof. Let $\mathcal{F}$ be the G-constellation corresponding to Λ. It is enough to consider submodules of $\mathcal{F}$ whose support is $\{l, l+1, \ldots, k\}$ for any $l, k \in I$. Let V be a submodule of $\mathcal{F}$ whose support is $\{l, l+1, \ldots, k\}$. Remember that $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ is considered as the dimension vector of the vertex simple at the vertex i. Note that

$$\begin{aligned} \theta(V) &= \theta(\varepsilon_l - \varepsilon_{l-1}) + \theta(\varepsilon_{l+1} - \varepsilon_l) \ldots + \theta(\varepsilon_k - \varepsilon_{k-1}) \\ &= \theta(\varepsilon_k - \varepsilon_{l-1}). \end{aligned}$$

Note that V is a submodule if and only if y_k and z_l are zero linear maps, i.e. $k \in \Lambda^y$ and $l \in \Lambda^z$. Therefore the assertion is proved. $\square$

Let ω be an element in S_r, $\mathfrak{C}(\omega)$ the Weyl chamber corresponding to ω, and

$\Delta(\omega)$ the set of simple roots, that is,

$$\Delta(\omega) := \left\{\varepsilon_{\omega(i)} - \varepsilon_{\omega(i-1)} \in \Phi \,\middle|\, i \in I, i \neq 0\right\},$$
$$\mathfrak{C}(\omega) := \left\{\theta \in \Theta \,\middle|\, \theta(\alpha) > 0 \quad \forall \alpha \in \Delta(\omega)\right\}.$$

Since $\mathcal{M}_\theta$ is irreducible for $\theta \in \mathfrak{C}(\omega)$ by [1,19], the number of θ-stable lacings should be r. We prove the following proposition by explicit calculations.

Proposition 5.1.6. *Let ω be an element in S_r, $\mathfrak{C}(\omega)$ the Weyl chamber corresponding to ω as above. Let θ be in the chamber $\mathfrak{C}(\omega)$. Then there exist exactly r θ-stable lacings. They are $\Lambda_j = (\Lambda_j^y, \Lambda_j^z)$ for each $1 \leq j \leq r$ where*

$$\begin{aligned} \Lambda_j^y &= \{\omega(j-1), \omega(j) \ldots, \omega(r-1)\}, \\ \Lambda_j^z &= \{\omega(0)+1, \omega(1)+1 \ldots, \omega(j-1)+1\}. \end{aligned} \tag{5.1.7}$$

Proof. First, from the definition of the chamber $\mathfrak{C}(\omega)$, note that $\theta\big(\varepsilon_{\omega(i)} - \varepsilon_{\omega(j)}\big) \geq 0$ if and only if $i \geq j$.

By Lemma 5.1.5, our Λ_j is θ-stable. It is enough to show that they are all θ-stable lacings.

Let $\Lambda = (\Lambda^y, \Lambda^z)$ be a θ-stable lacing.

Suppose $\omega(0)$ is in Λ^y. By Lemma 5.1.5, only $\omega(0)+1$ can be in Λ^z and hence from the condition (ii) in Definition 5.1.2, $\Lambda^y = I$. The number of elements in Λ^y must be one, so $\Lambda^z = \{\omega(0)+1\}$. Therefore $\Lambda = \Lambda_1$.

Suppose j is the minimum such that $\omega(j-1)$ is in Λ^y. By Lemma 5.1.5, only $\omega(0)+1, \omega(1)+1, \ldots, \omega(j-1)+1$ can be in Λ^z. Hence

$$\mid \Lambda^y \mid \leq r - j + 1 \quad \text{and} \quad \mid \Lambda^z \mid \leq j.$$

Since $\mid \Lambda^y \mid + \mid \Lambda^z \mid = r+1$, we have

$$\Lambda^y = \{\omega(j-1), \omega(j) \ldots, \omega(r-1)\},$$
$$\Lambda^z = \{\omega(0)+1, \omega(1)+1 \ldots, \omega(j-1)+1\},$$

i.e. $\Lambda = \Lambda_j$. $\square$

Observe that when we move from Λ_j to Λ_{j+1}, we add $\omega(j)+1$ to Λ^z and remove $\omega(j-1)$ from Λ^y.

Remark 5.1.8. Each Λ_j corresponds to a torus fixed point in $\mathcal{M}_\theta$. ♦

We now describe a local chart of $\mathcal{M}_\theta$ containing the G-constellation corresponding to a θ-stable lacing Λ. Assume that θ is generic in the Weyl chamber $\mathfrak{C}(\omega)$ for a permutation $\omega \in S_r$.

Let $\Lambda = \Lambda_j = (\Lambda_j^y, \Lambda_j^z)$ be the θ-stable lacing in Proposition 5.1.6:

$$\begin{aligned} \Lambda_j^y &= \{\omega(j-1), \omega(j) \dots, \omega(r-1)\}, \\ \Lambda_j^z &= \{\omega(0)+1, \omega(1)+1 \dots, \omega(j-1)+1\}. \end{aligned}$$

As is described above, Λ encodes which linear maps (or y, z-actions) vanish. After changing basis, setting

$$\begin{cases} y_i = 1 & \text{if } i \notin \Lambda^y, \\ z_i = 1 & \text{if } i \notin \Lambda^z, \end{cases}$$

gives a local chart S_j of $\mathcal{M}_\theta$. Set coordinates η_j, ζ_j to be

$$\begin{cases} \eta_j &= y_{\omega(j-1)}, \\ \zeta_j &= z_{\omega(j-1)+1}. \end{cases} \tag{5.1.9}$$

From the commutation relations, it follows that

$$y_0 z_1 = y_1 z_2 = \dots = y_{r-1} z_0 = \eta_j \zeta_j.$$

Note that for each $i \neq \omega(j-1)$ either $i \notin \Lambda^y$ or $i+1 \notin \Lambda^z$. This means that for each $i \neq \omega(j-1)$ either y_i or z_{i+1} is set to be 1. Thus

$$\begin{cases} z_i = \eta_j \zeta_j & \text{if } y_i = 1, \quad \text{i.e. } i \notin \Lambda^y, \\ y_i = \eta_j \zeta_j & \text{if } z_{i+1} = 1, \text{ i.e. } i+1 \notin \Lambda^z. \end{cases}$$

Therefore the affine open set S_j of $\mathcal{M}_\theta$ is isomorphic to $\mathbb{C}^2$ with the coordinates η_j, ζ_j.

We have commutative diagrams when $\eta_j \neq 0$ and $\zeta_{j+1} \neq 0$:

$$\begin{tikzcd}
\Lambda_j : & \mathbb{C}e_{\omega(j)} \arrow[r, shift left, "\eta_j\zeta_j"] \arrow[d, "1"'] & \mathbb{C}e_{\omega(j)+1} \arrow[l, shift left, "1"] \arrow[d, "\eta_j"] & \mathbb{C}e_{\omega(j-1)} \arrow[r, shift left, "\eta_j"] \arrow[d, "\eta_j"'] & \mathbb{C}e_{\omega(j-1)+1} \arrow[l, shift left, "\zeta_j"] \arrow[d, "1"] \\
\Lambda_{j+1} : & \mathbb{C}e_{\omega(j)} \arrow[r, shift left, "\eta_{j+1}"] & \mathbb{C}e_{\omega(j)+1} \arrow[l, shift left, "\zeta_{j+1}"] & \mathbb{C}e_{\omega(j-1)} \arrow[r, shift left, "1"] & \mathbb{C}e_{\omega(j-1)+1} \arrow[l, shift left, "\eta_{j+1}\zeta_{j+1}"]
\end{tikzcd}$$

where the going right (resp. going left) arrows are y-actions (resp. z-actions) and the going down arrows mean changing basis. From this diagram, one can see that the gluing of two affine pieces S_j and S_{j+1} is given by

$$\begin{array}{ccc} S_j \setminus (\eta_j = 0) & \longrightarrow & S_{j+1} \setminus (\zeta_{j+1} = 0) \\ (\eta_j, \zeta_j) & \longmapsto & (\eta_j^2 \zeta_j, \eta_j^{-1}). \end{array}$$

Observe that there is a divisor $E_j \cong \mathbb{P}^1$ in $S_j \cup S_{j+1}$, which is the coordinate axis of $\eta_j = \zeta_{j+1}^{-1}$. Note that the divisor E_j is given by $\zeta_j = 0$ in S_j and it is given by $\eta_{j+1} = 0$ in S_{j+1}. Since $\eta_{j+1} = \eta_j^2 \zeta_j$, the divisor E_j is a (-2)-curve.

Let S be the union of S_j's with the gluing above. As the Λ_j's are all possible lacings, $\cup S_j$ forms an affine open cover of $\mathcal{M}_\theta$ and hence S is isomorphic to $\mathcal{M}_\theta$. We saw that S contains (-2)-curves $E_1, \dots, E_{r-1}$.

The following theorem is called *the McKay correspondence* (see [1, 19]).

Theorem 5.1.10 (the McKay correspondence)**.** *Let G be the group of type $\frac{1}{r}(1, -1)$. For any generic parameter θ, the moduli space $\mathcal{M}_\theta$ is the minimal resolution of $\mathbb{C}^2/G$.*

5.1.3 Universal families and intersection numbers

By [18], if $\theta \in \mathfrak{C}(\omega)$, then $\mathcal{M}_\theta$ is the fine moduli space of θ-stable G-constellations. Thus the moduli space $\mathcal{M}_\theta$ accompanies the universal family $\mathcal{L}$ which can be decomposed as

$$\mathcal{L} = \bigoplus_{\rho_i \in \mathrm{Irr}(G)} \mathcal{L}_i \otimes \rho_i.$$

Each direct summand $\mathcal{L}_i$ is a locally free sheaf of rank one. We call $\mathcal{L}_i$ the tautological line bundle of ρ_i.

Let L be the lattice

$$L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r}(1, -1).$$

For each $0 \leq i \leq r$, let v_i denote the lattice point

$$v_i = \frac{1}{r}(i, r - i)$$

of L. For each $1 \leq j \leq r$, define σ_j to be the cone

$$\sigma_j := \mathrm{Cone}(v_{j-1}, v_j).$$

Let Σ be the fan consisting of the σ_j's and their faces. Note that the associated toric variety X_Σ is smooth and that X_Σ is the minimal resolution of the quotient variety $X = \mathbb{C}^2/G$.

Let E_i denote the irreducible exceptional divisor corresponding to the ray generated by v_i for $1 \le i \le r-1$. Then the $[E_i]$'s form a basis of the homology group $\mathrm{H}_2(X_\Sigma, \mathbb{Z})$, i.e.

$$\mathrm{H}_2(X_\Sigma, \mathbb{Z}) = \bigoplus_{1 \le i \le r-1} \mathbb{Z}[E_i].$$

It is well known that if the stability parameter θ is in Θ_+, the first Chern classes $c_1(\mathcal{L}_i)$ of the tautological line bundles $\mathcal{L}_i$ form the dual basis to $[E_i]$ [10].

For a generic GIT parameter $\theta \in \Theta$, from GIT, it is known that

$$\mathcal{L}_\theta := \bigotimes_{i \in I} \mathcal{L}_i^{\theta(\rho_i)}$$

is relatively ample over the variety $X = \mathcal{M}_0 = \mathbb{C}^2/G$.

We now show this with an explicit calculation without GIT.

Let ω be a permutation and $\mathfrak{C}(\omega)$ the corresponding (open) Weyl chamber. As we did in (5.1.1), let $\mathbf{w}_i$ be the rays of $\mathfrak{C}(\omega)$ which form the dual basis to the simple roots $\Delta(\omega)$, i.e.

$$\mathbf{w}_i = \sum_{j=0}^{i-1} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right)$$

for $i = 1, 2, \ldots, r-1$, where $\{\theta_i\}_{i=1}^{r-1}$ is the dual basis with respect to $\{\alpha_i\}_{i=1}^{r-1}$.

Proposition 5.1.11. *With the notation as above, let $\mathcal{L}_k$ be the tautological line bundle of ρ_k. Define the line bundle*

$$\mathcal{F}_i := \bigotimes_{k \in I} \mathcal{L}_k^{\mathbf{w}_i(\rho_k)}.$$

for any $i = 1, 2, \ldots, r-1$. Then $c_1\big(\mathcal{F}_i\big) \cdot E_j = \delta_{ij}$, i.e. $\{\mathcal{F}_i\}_{i=1}^{r-1}$ is the dual basis to $\{[E_i]\}$. Therefore, the (fractional) line bundle $\mathcal{L}_\theta$ is relatively ample over $X = \mathcal{M}_0 = \mathbb{C}^2/G$ for any generic parameter $\theta \in \mathfrak{C}(\omega)$.

Proof. Let Λ_j be the jth lacing and Γ_j the corresponding G-iraffe. As in (5.1.9), the following two parameters play as the coordinates of the affine open set $S_j = U(\Gamma_j)$:

$$\begin{cases} \eta_j &= y_{\omega(j-1)}, \\ \zeta_j &= z_{\omega(j-1)+1}. \end{cases}$$

Consider $\mathcal{F}_1 = \mathcal{L}_{\omega(0)+1} \otimes \mathcal{L}_{\omega(0)}^{-1}$. By the construction, the lacing $\Lambda_1 = (\Lambda_1^y, \Lambda_1^z)$ is

$$\begin{aligned} \Lambda_1^y &= \{\omega(0), \omega(1), \ldots, \omega(r-1)\}\,, \\ \Lambda_1^z &= \{\omega(0)+1\}\,. \end{aligned}$$

Observe that Λ_j^y does not contain $\omega(0)$ for $j > 1$. Note that the line bundle $\mathcal{F}_1$ corresponds to the linear map

$$y_{\omega(0)} \colon \rho_{\omega(0)} \to \rho_{\omega(0)+1}.$$

From the lacings, one can see that

$$y_{\omega(0)} = \begin{cases} \eta_1 & \text{if } j = 1, \\ 1 & \text{otherwise} \end{cases}$$

on each open set S_j. Since the divisor E_j is the coordinate axis of η_j, one can see that

$$c_1(\mathcal{F}_1) \cdot E_j = \begin{cases} 1 & \text{if } j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, one can show this in terms of G-iraffes. By construction of the corresponding G-iraffes, $\mathcal{F}_1$ is the line bundle defined by z^{r-1} on the open set S_1 and $\mathcal{F}_1$ is the line bundle defined by y on the open set S_j for $j > 1$. Since the exceptional divisor $E_j \cong \mathbb{P}^1$ is defined by the ratio $[z^{r-j} : y^j]$, we have the same result.

Since $\mathbf{w}_{i+1} - \mathbf{w}_i = \theta_{\omega(i)+1} - \theta_{\omega(i)}$,

$$\mathcal{F}_{i+1} \otimes \mathcal{F}_i^{-1} = \mathcal{L}_{\omega(i)+1} \otimes \mathcal{L}_{\omega(i)}^{-1}.$$

If we show that

$$c_1\big(\mathcal{L}_{\omega(i)+1} \otimes \mathcal{L}_{\omega(i)}^{-1}\big) \cdot E_j = \begin{cases} 1 & \text{if } j = i+1, \\ -1 & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases} \tag{5.1.12}$$

then it follows that $c_1\big(\mathcal{F}_i\big) \cdot E_j = \delta_{ij}$ from induction on i.

Fix i such that $1 < i < r$. Consider $\mathcal{F}_i = \mathcal{L}_{\omega(i)+1} \otimes \mathcal{L}_{\omega(i)}^{-1}$, which corresponds to the linear map $y_{\omega(i)}$. Note that

$$y_{\omega(i)} = \begin{cases} \eta_j \zeta_j & \text{if } j \le i, \\ \eta_{i+1} & \text{if } j = i+1, \\ 1 & j > i+1, \end{cases}$$

on each open set S_j. Since the divisor E_j is the coordinate axis of η_j, from the fact that $\eta_j \zeta_j = \eta_{j+1}\zeta_{j+1}$ and calculations of transition functions, the claim (5.1.12) follows. Alternatively, one can show this in terms of G-iraffes. By construction of corresponding G-iraffes, one can see that $\mathcal{F}_i$ is the line bundle defined by z^{-1} on the open set S_j for $j \le i$, that $\mathcal{F}_i$ is defined by $\frac{z^{r-i}}{y^{i-1}}$ on the open set S_i, and that $\mathcal{F}_i$ is define by y on the open set S_j for $j > i+1$. Since the exceptional divisor $E_j \cong \mathbb{P}^1$ is defined by the ratio $[z^{r-j} : y^j]$, we have proved the claim.

The relative ampleness of $\mathcal{L}_\theta$ follows from the fact that θ is a strictly positive linear combination of $\mathbf{w}_i$'s. □

5.1.4 Example: type $\frac{1}{7}(1,6)$

This section calculates lacings for the finite group G of type $\frac{1}{7}(1,6)$ with a fixed Weyl chamber of A_6. These lacings give an affine cover of the moduli space of θ-stable G-constellations. In addition, we present the intersection matrix between the universal family of the moduli space and the exceptional divisors.

Let $G \subset \mathrm{SL}_2(\mathbb{C})$ be the finite group of type $\frac{1}{7}(1,-1)$. Its McKay quiver is shown in Figure 5.1.1.

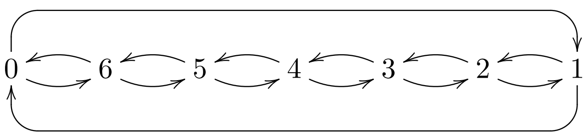


Figure 5.1.1: McKay quiver for G of type $\frac{1}{7}(1,6)$

In Figure 5.1.1, the number i denotes the vertex corresponding to ρ_i and the upper (resp. lower) curved arrows correspond to y-actions (resp. z-actions).

Let ω be the permutation of $I := \{0, 1, \dots, 5, 6\}$ given by

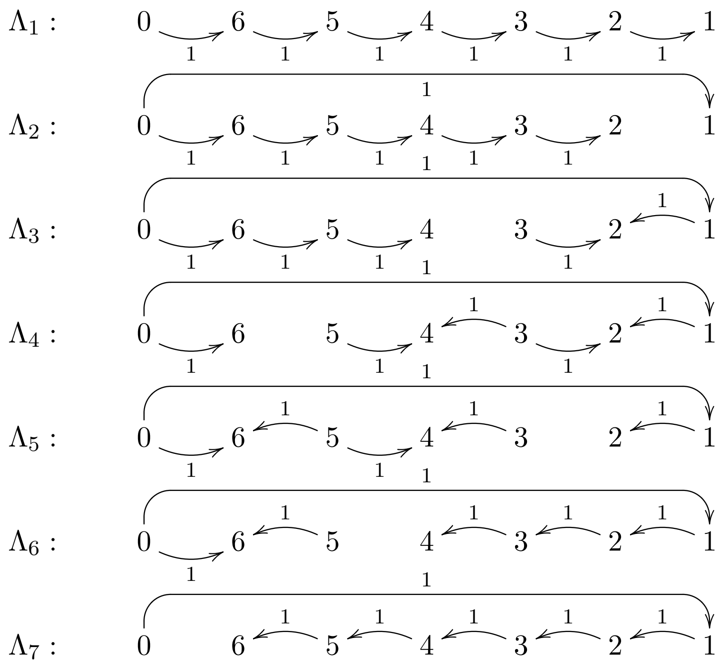
$$\omega = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}.$$

Note that the corresponding set of simple roots is

$$\begin{aligned}\Delta(\omega) &= \{\varepsilon_1 - \varepsilon_0, \varepsilon_3 - \varepsilon_1, \varepsilon_5 - \varepsilon_3, \varepsilon_2 - \varepsilon_5, \varepsilon_4 - \varepsilon_2, \varepsilon_6 - \varepsilon_4\} \\ &= \{\alpha_1, \alpha_2 + \alpha_3, \alpha_4 + \alpha_5, -\alpha_3 - \alpha_4 - \alpha_5, \alpha_3 + \alpha_4, \alpha_5 + \alpha_6\},\end{aligned}$$

and that we have a Weyl chamber $\mathfrak{C}(\omega)$ corresponding to ω, which forms a chamber in the GIT stability parameter space of G-constellations.

According to Proposition 5.1.6, we have 7 lacings which give the following 7 torus invariant G-constellations:



where marked linear maps are set to be 1.

Observe that the difference between Λ^j and Λ^{j+1} is: we have one more nonzero y-arrow and we have one less nonzero z-arrow. We call this process "cutting and adding laces".

As is described above, these lacings give local charts. For example, consider Λ_4 and set two linear maps as the coordinates

$$\begin{cases} \eta &= y_5, \\ \zeta &= z_6. \end{cases}$$

One can see that the point $(\eta, \zeta) \in \mathbb{C}^2$ corresponds to the following G-constellation:

$$\begin{tikzcd}
0 \arrow[r, bend right, "1"'] \arrow[rrrrrr, bend left=60, "1"] & 6 \arrow[l, bend right, "\eta\zeta"'] \arrow[r, bend right, "\eta\zeta"'] & 5 \arrow[l, bend right, "\eta\zeta"'] \arrow[r, bend right, "1"'] & 4 \arrow[l, bend right, "\eta\zeta"'] \arrow[r, bend right, "\eta\zeta"'] & 3 \arrow[l, bend right, "1"'] \arrow[r, bend right, "\eta\zeta"'] & 2 \arrow[l, bend right, "\eta\zeta"'] \arrow[r, bend right, "\eta\zeta"'] & 1 \arrow[l, bend right, "1"'] \arrow[llllll, bend left=60, "\eta\zeta"]
\end{tikzcd}$$

where $(\eta, \zeta) = (0, 0)$ corresponds to the torus invariant G-constellation defined by Λ_4.

For each Λ_j, there exists a unique G-iraffe Γ_j (see Proposition 2.6.7):

$$\begin{aligned}
\Gamma_1 &= \{1,\ z,\ z^2,\ z^3,\ z^4,\ z^5,\ z^6\},\\
\Gamma_2 &= \{1,\ z,\ z^2,\ z^3,\ z^4,\ z^5,\ y\},\\
\Gamma_3 &= \{1,\ z,\ z^2,\ z^3,\ \tfrac{y^2}{z},\ y^2,\ y\},\\
\Gamma_4 &= \{1,\ z,\ \tfrac{y^3}{z^2},\ \tfrac{y^3}{z},\ \tfrac{y^2}{z},\ y^2,\ y\},\\
\Gamma_5 &= \{1,\ z,\ \tfrac{z}{y},\ \tfrac{z^2}{y},\ \tfrac{z^2}{y^2},\ y^2,\ y\},\\
\Gamma_6 &= \{1,\ z,\ \tfrac{z}{y},\ y^4,\ y^3,\ y^2,\ y\},\\
\Gamma_7 &= \{1,\ y^6,\ y^5,\ y^4,\ y^3,\ y^2,\ y\}.
\end{aligned}$$

With the lattice $L = \mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{7}(1, -1)$, one can show that each G-iraffe Γ_j satisfies $U(\Gamma_j) = \operatorname{Spec} \mathbb{C}[S(\Gamma_j)] = D(\Gamma_j) = U_{\sigma_j}$, where the toric cone $\sigma_j \subset L_{\mathbb{R}}$ is

$$\sigma_j = \operatorname{Cone}\left(\tfrac{1}{7}(j-1, r-j+1), \tfrac{1}{7}(j, r-j)\right).$$

Moreover, the (-2)-curve E_j is the corresponding divisor to the ray $v_j := \frac{1}{7}(j, r-j)$.

Let $\mathcal{L}_i$ be the tautological line bundle of ρ_i, which is a direct summand of the universal family $\bigoplus_{i \in I} \mathcal{L}_i$. As is stated in Section 4.4, over the toric affine open set $U(\Gamma_j)$, the line bundle $\mathcal{L}_i$ is defined by the element of weight i in Γ_j.

We calculate the intersection number $c_1(\mathcal{L}_i) \cdot E_j$. For example, consider $c_1(\mathcal{L}_5)$ and note that

$$c_1(\mathcal{L}_5) \cdot E_j = \begin{cases} 0 & \text{if } j = 1,\\ 0 & \text{if } j = 2,\\ 1 & \text{if } j = 3,\\ -1 & \text{if } j = 4,\\ 0 & \text{if } j = 5,\\ 1 & \text{if } j = 6.\end{cases}$$

The intersection matrix $(c_1(\mathcal{L}_i) \cdot E_j)_{i,j}$ is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

whose inverse matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.1.13}$$

On the other hand, by (5.1.1), the open Weyl chamber $\mathfrak{C}(\omega)$ associated to the permutation ω is the cone generated by the row vectors of the following matrix:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with the basis $\{\theta_i\}$. One can see that the submatrix obtained by deleting the first column is the same as the matrix (5.1.13).

5.2 Chamber structures and elephants

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{r}(b, 1, -1)$, with b coprime to r, which is the same group as before but taking another primitive rth root of unity. In this section, we investigate the chamber structure of the GIT parameter space of G-constellations.

Let ρ_i be the irreducible representation of G whose weight is i. We can identify $I := \mathrm{Irr}(G)$ with $\mathbb{Z}/r\mathbb{Z}$.

Recall the McKay quiver of G is the quiver whose vertex set is I with the $3r$

following arrows:

$$x_i\colon i \to i+b,$$
$$y_i\colon i \to i+1,$$
$$z_i\colon i \to i-1,$$

for each $i \in I$. The representation of the McKay quiver of G with commutation relations is the representation of the McKay quiver whose dimension vector is $(1,\dots,1)$ satisfying the following relations:

$$\begin{cases} x_i y_{i+b} = y_i x_{i+1}, \\ x_i z_{i+b} = z_i x_{i-1}, \\ y_i z_{i+1} = z_i y_{i-1}. \end{cases}$$

Let $A \subset \mathrm{SL}_2(\mathbb{C})$ be of type $\frac{1}{r}(1,-1)$ with coordinates y, z. The McKay quiver of A is the quiver whose vertex set is I with the $2r$ following arrows:

$$y_i\colon i \to i+1,$$
$$z_i\colon i \to i-1,$$

for each $i \in I$. The representation of the McKay quiver of A with commutation relations is the representation of the McKay quiver whose dimension vector is $(1,\dots,1)$ satisfying the following relations:

$$y_i z_{i+1} = z_i y_{i-1} \quad \text{for all } i \in I.$$

Note that the GIT parameter space Θ of G-constellations can be identified with

$$\Theta = \left\{ \theta = (\theta^i) \in \mathbb{Q}^r \;\middle|\; \sum \theta^i = 0 \right\},$$

which is also the GIT parameter space of A-constellations. Furthermore, we have the following proposition.

Proposition 5.2.1. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{r}(b,1,-1)$ and $A \subset \mathrm{SL}_2(\mathbb{C})$ the finite subgroup of type $\frac{1}{r}(1,-1)$. Then the chamber structure of the GIT parameter space of G-constellations is finer than, or equal to, that of A-constellations.*

Proof. It suffices to show that a wall of the GIT parameter space of A-constellations is also a wall of the GIT parameter space of G-constellations.

Let θ be a stability parameter on a wall of the GIT parameter space of A-constellations. This means that there exists a θ-semistable A-constellation $\mathcal{F}$ such that it is not θ-stable, i.e. there exists a $\mathbb{C}[y,z]$-submodule $\mathcal{G}$ with $\theta(\mathcal{G}) = 0$.

Note that we have a natural identification between A-constellations and G-constellations whose x-action is zero. Thus $\mathcal{F}$ can be thought of as a G-constellation and $\mathcal{G}$ is a $\mathbb{C}[x,y,z]$-submodule of $\mathcal{F}$ with $\theta(\mathcal{G}) = 0$. As it is easy to see that $\mathcal{F}$ is θ-semistable G-constellation, it proves that θ is also on a wall of the GIT parameter space of G-constellations. $\square$

Note that the chamber structure of GIT parameter space of A-constellations is the same as the Weyl chamber structure of A_{r-1}.

Conjecture 5.2.2. *The chamber structure of the GIT stability parameter space Θ of G-constellations coincides with the Weyl chamber structure of A_{r-1}.*

Let θ be a generic element of the GIT parameter space of G-constellations. By Proposition 5.2.1, θ is generic in the GIT parameter space of A-constellations so there exists an open Weyl chamber $\mathfrak{C}$ such that $\theta \in \mathfrak{C}$. Let ω be the corresponding element in S_r as in Section 5.1.

Let us consider the space of G-constellations $\operatorname{Rep} G$ and the space of A-constellations $\operatorname{Rep} A$. Consider the reductive group

$$\mathrm{GL}(\delta) := \prod_{i \in I} \mathbb{C}^{\times}$$

acting on $\operatorname{Rep} G$ and $\operatorname{Rep} A$ as basis change. The moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations is

$$\mathcal{M}_\theta = \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[\operatorname{Rep} G]_{\chi_\theta^n} \right).$$

Let $\operatorname{Rep}^s G$ be the θ-stable locus in $\operatorname{Rep} G$ and $\operatorname{Rep}^s A$ the θ-stable locus in $\operatorname{Rep} A$. We can identify $\operatorname{Rep} A$ with the closed subvariety of $\operatorname{Rep} G$ defined by $x_0 = \cdots = x_{r-1} = 0$ and $\operatorname{Rep}^s A$ with the closed subvariety $\widetilde{S_\theta}$ of $\operatorname{Rep}^s G$ defined by $x_0 = \cdots = x_{r-1} = 0$.

Since $\widetilde{S_\theta}$ is a $\mathrm{GL}(\delta)$-invariant closed set, and $\mathcal{M}_\theta$ is a geometric quotient, the inclusion $\widetilde{S_\theta} \subset \operatorname{Rep}^s G$ induces an inclusion $S_\theta \subset \mathcal{M}_\theta$

$$\begin{tikzcd} \widetilde{S_\theta} \arrow[r, hook] \arrow[d] & \operatorname{Rep}^s G \arrow[d] \\ S_\theta \arrow[r, hook] & \mathcal{M}_\theta \end{tikzcd}$$

where S_θ is the closed subvariety of $\mathcal{M}_\theta$ parametrising G-constellations on which x

acts trivially. Note that the variety S_θ is isomorphic to the moduli space of θ-stable A-constellations.

Remark 5.2.3. By Proposition 5.1.6, S_θ has r torus invariant points which represent torus invariant θ-stable G-constellations. ♦

Let D be the hyperplane section of $\mathbb{C}^3/G$ defined by $x = 0$. Then D is isomorphic to $\mathbb{C}^2/A$ and has an A_{r-1} singularity as in Section 3.4. Since $\mathcal{M}_0$ is isomorphic to $\mathbb{C}^3/G$ by Proposition A.0.1, we have the following diagram

$$\begin{tikzcd} S_\theta \arrow[r, hook, "\text{codim.1}"'] \arrow[d] & Y_\theta \arrow[r, hook, "\text{irr.}"'] \arrow[d] & \mathcal{M}_\theta \arrow[d] \\ D \arrow[r, hook, "\text{codim.1}"'] & \mathbb{C}^3/G \arrow[r, equal] & \mathcal{M}_0 \end{tikzcd}$$

where the vertical morphisms are the canonical projective morphisms induced by GIT quotients. As is known, the morphism $S_\theta \to D$ is the minimal resolution of D.

5.3 Irreducibility for type $\frac{1}{2k+1}(1, 2, 2k-1)$

In this section, we prove Conjecture 4.3.3 for the group of type $\frac{1}{2k+1}(1, 2, 2k-1)$ with a parameter θ in the admissible GIT chamber $\mathfrak{C}_a \subset \Theta$. This can be proved by finding all θ-stable torus invariant G-constellations for $\theta \in \mathfrak{C}_a$.

Throughout this section, let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{2k+1}(k+1, 1, 2k)$, which is the same type as the type of $\frac{1}{2k+1}(1, 2, 2k-1)$. Thus $r = 2k+1$, $a = 2$, and set $b := k+1$.

Warning 5.3.1. Throughout this section, we consider the finite subgroup of type $\frac{1}{2k+1}(k+1, 1, 2k)$ so that the weight of y is 1 and that the weight of z is $2k$. The results in previous sections can be easily transferred to this notation. ♦

5.3.1 G-iraffes

In this section, we present all G-iraffes using the method in Section 4.1.

Consider the lattice $L = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{r}(1, 2, r-2)$. For each $0 \le i \le r$, let v_i be the lattice point $\frac{1}{r}(\overline{ib}, i, r-i)$. Consider the toric fan Σ of the economic resolution

Y of $X = \mathbb{C}^3/G$. In the fan Σ, we have the following $2r-1$ full dimensional cones:

$$\begin{cases} \sigma_i = \mathrm{Cone}(e_1, v_{r-i+1}, v_{r-i}) & \text{for } 1 \leq i \leq r, \\ \sigma_i^{\triangle} = \mathrm{Cone}(v_{2i-1}, v_{2i-2}, v_{2i}) & \text{for } 1 \leq i \leq k, \\ \sigma_i^{\bigtriangledown} = \mathrm{Cone}(e_2, v_{2i-2}, v_{2i}) & \text{for } 1 \leq i \leq k. \end{cases}$$

Proposition 5.3.2. *With the notation as above, let Γ_l, $\Gamma_i^{\triangle}$, and $\Gamma_i^{\bigtriangledown}$ be the G-iraffes corresponding to the cones σ_l, $\sigma_i^{\triangle}$, and $\sigma_i^{\bigtriangledown}$, respectively. Then the following hold:*

(i) $\Gamma_1 = \{1, z, z^2, \ldots, z^{2k}\}$.

(ii) $\Gamma_l = \left\{ \begin{matrix} 1, z, \ldots, z^{k-i}, y, \ldots, y^{i-1}, y^i \\ \frac{y^{i+1}}{z^{k-i}}, \frac{y^{i+1}}{z^{k-i-1}}, \ldots, y^{i+1}, \frac{y^{i+2}}{z^{k-i}}, \ldots, \frac{y^{l-1}}{z^{k-i}} \end{matrix} \right\}$, *if $l = 2i+1$ is odd.*

(iii) $\Gamma_l = \left\{ \begin{matrix} 1, z, \ldots, z^{k-i}, y, \ldots, y^{i-1}, y^i \\ \frac{z^{k-i+1}}{y^{i-1}}, \frac{z^{k-i+1}}{y^{i-2}}, \ldots, z^{k-i+1}, \frac{z^{k-i+2}}{y^{i-1}}, \ldots, \frac{z^{2k-2i+1}}{y^{i-1}} \end{matrix} \right\}$, *if $l = 2i$ is even.*

(iv) $\Gamma_i^{\triangle} = \left\{ \begin{matrix} 1, & z, & \ldots, & z^{i-1}, & y, & \ldots, & y^{k-i-1}, & y^{k-i} \\ x, & xz, & \ldots, & xz^{i-1}, & xy, & \ldots, & xy^{k-i-1} & \end{matrix} \right\}$.

(v) $\Gamma_i^{\bigtriangledown} = \left\{ \begin{matrix} 1, & z, & \ldots, & z^{i-1}, & x^2, & \ldots, & x^{2k-2i+1}, & x^{2k-2i+2} \\ x, & xz, & \ldots, & xz^{i-1} & & & & \end{matrix} \right\}$.

Proof. For $k = 1$, G-iraffes are in Example 4.1.3. Using round down functions, induction on k proves the assertion. □

Remark 5.3.3. Note that the G-iraffes $\Gamma_i^{\triangle}$ and $\Gamma_i^{\bigtriangledown}$ are Nakamura G-graphs. ♦

5.3.2 The admissible chamber

In this section, we explicitly express the admissible chamber for the group G of type $\frac{1}{2k+1}(k+1, 1, 2k)$. We recall the identification (4.2.7) and (4.2.8).

Let

$$\{\varepsilon_l^L \mid l = 0, 1\}, \quad \{\varepsilon_k^R \mid k = 0, 1, \ldots, 2k-2\}$$

be the standard basis of $\mathbb{Q}^2$ and $\mathbb{Q}^{2k-1}$, respectively. Assume that Δ^L and Δ^R are the admissible set of simple roots for type $\frac{1}{2}(1,1,1)$ and $\frac{1}{2k-1}(k, 1, 2k-2)$. Let the standard basis $\{\varepsilon_i \,|\, i \in I\}$ of $\mathbb{Q}^{2k+1}$ be identified with the union of the two sets

$$\{\varepsilon_l^L \,|\, l = 0, 1\} \text{ and } \{\varepsilon_k^R \,|\, k = 0, 1, \ldots, 2k-2\}$$

using the following identification:

$$\begin{aligned} \varepsilon_l^L &= \varepsilon_i \quad \text{with } i = \left\lceil \tfrac{r(l+1)}{2} \right\rceil - 1, \quad \text{for } r-2 \le i < r, \\ \varepsilon_j^R &= \varepsilon_i \quad \text{with } i = \left\lfloor \tfrac{rj}{r-a} \right\rfloor, \quad \text{for } 0 \le i < r-2. \end{aligned} \tag{5.3.4}$$

With this identification, the admissible set Δ of simple roots is

$$\Delta = \Delta^L \cup \{\varepsilon_{\lfloor \frac{r}{2} \rfloor} - \varepsilon_{r - \lceil \frac{r}{r-2} \rceil}\} \cup \Delta^R. \tag{5.3.5}$$

Remember that the the root $\alpha_i = \varepsilon_i - \varepsilon_{i-1}$ are considered as the dimension vectors for the vertex simple at i of the McKay quiver representations.

Proposition 5.3.6. *For the group of type $\frac{1}{2k+1}(k+1, 1, 2k)$, the admissible set Δ_a of simple roots is*

$$\Delta_a = \{\varepsilon_{2k} - \varepsilon_k, \varepsilon_k - \varepsilon_{2k-1}, \varepsilon_{2k-1} - \varepsilon_{k-1}, \ldots \varepsilon_{k+1} - \varepsilon_1, \varepsilon_1 - \varepsilon_0\}$$

and the corresponding permutation ω is

$$\omega = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \ldots & 2k-2 & 2k-1 & 2k \\ 0 & 1 & k+1 & 2 & k+2 & \ldots & 2k-1 & k & 2k \end{pmatrix}, \tag{5.3.7}$$

i.e.

$$\omega(l) = \begin{cases} 0 & \text{if } l = 0, \\ \frac{l+1}{2} & \text{if } l \text{ is odd}, \\ k + \frac{l}{2} & \text{otherwise}, \end{cases}$$

for $l \in I = \{0, 1, 2, \ldots, 2k-1, 2k\}$.

Proof. Induction on k. □

Remark 5.3.8. Note that the ith ray of the admissible chamber is

$$\sum_{j=0}^{i-1} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right), \tag{5.3.9}$$

where $\{\theta_i\}_{i=1}^{r-1}$ is the dual basis with respect to $\{\alpha_i\}_{i=1}^{r-1}$. ♦

5.3.3 Torus invariant G-constellations

Let θ be a generic parameter in the admissible chamber $\mathfrak{C}_a$. Let $\mathcal{F}$ be a θ-stable torus invariant G-constellation. Let x_i, y_i, z_i denote the action of x, y, z on the vector space $\mathbb{C}\rho_i$, respectively.

Recall Lemma 2.6.4 says that if $\mathcal{F}$ is a torus invariant G-constellation, then there is no defined (undirected) cycle of type $\mathbf{m}$ with $\mathbf{m} \neq \mathbf{1}$.

Remark 5.3.10. Since yz is a G-invariant monomial, any path induced by yz in any torus invariant G-constellation $\mathcal{F}$ is zero. In other words, if y_i is nonzero in $\mathcal{F}$, then z_{i+1} is zero; if z_i is nonzero in $\mathcal{F}$, then y_{i-1} is zero. ♦

We have two cases: (1) $x_0 = 0$: (2) $x_0 \neq 0$.

Case $x_0 = 0$.

Assume that $x_0 = 0$, i.e. x acts on $\mathbb{C}\rho_0$ trivially. In this case, if we prove that $x_i = 0$ for all i, then it follows that the G-constellation $\mathcal{F}$ is in the list of Proposition 5.3.2 from the discussion in Section 5.1. We now prove that we have at most r θ-stable torus invariant G-constellations.

1st case: Suppose that $y_0 = 0$. From (5.3.9), note that the first ray of the admissible chamber is

$$\sum_{j=0}^{0} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right) = \theta_1 - \theta_0,$$

which means that there exists a nonzero path from ρ_0 to ρ_1 in $\mathcal{F}$. However, since $x_0 = y_0 = 0$, the only possible nonzero path from ρ_0 to ρ_1 is induced by z^{2k}. Thus for each ρ_i, we have a nonzero path from ρ_0 to ρ_i induced by z^{2k-i}. From this and Remark 2.6.3, it follows that $x_i = y_i = 0$ in $\mathcal{F}$ for all i. Therefore $\mathcal{F}$ is the G-constellation corresponding to Γ_1 in the list of Proposition 5.3.2.

2nd case: Suppose that $y_0 \neq 0$ and $y_1 = 0$. From (5.3.9), note that the second ray of the admissible chamber is

$$\sum_{j=0}^{1} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right) = \theta_1 - \theta_0 + \theta_2 - \theta_1 = \theta_2 - \theta_0,$$

which means that there exists a nonzero path $\mathbf{p}$ from ρ_0 to ρ_2 in $\mathcal{F}$. Suppose that the path $\mathbf{p}$ is induced by a monomial $\mathbf{m} = x^\alpha y^\beta z^\gamma$. From $x_0 = 0$, it follows that $\alpha = 0$.

By Remark 5.3.10, one can see that either β or γ is zero. In fact, if $\gamma = 0$, then the path $\mathbf{p}$ is induced by y^2 so $\mathbf{p} = y_0 y_1$ is nonzero, which contradicts the assumption $y_1 = 0$. Thus $\mathbf{p}$ is induced by z^{2k-1}. One can show that $\mathcal{F}$ is the G-constellation corresponding to Γ_2 in the list of Proposition 5.3.2.

$(l+1)$**th case:** Suppose that $y_{\omega(0)}, y_{\omega(1)}, \ldots, y_{\omega(l-1)} \neq 0$ and $y_{\omega(l)} = 0$ for $2 \leq l \leq 2k-2$. From $x_0 = 0$ and $y_0 \neq 0$, we have $x_1 = 0$ because $y_0 x_1 = x_0 y_{k+1} = 0$. From (5.3.9), note that the $(l+1)$th ray of the admissible chamber is

$$\sum_{j=0}^{l} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right) = \theta_{\omega(l)+1} + \theta_{\omega(l-1)+1} - \theta_{k+1} - \theta_0.$$

We have two cases: (A) l is odd: (B) l is even:

Case (A) $3 \leq l$ **is odd.** In this case, $\omega(l) = \frac{l+1}{2}$ and $\omega(l-1) = k + \frac{l-1}{2}$. Thus we have $y_{\frac{l+1}{2}} = 0$ and $y_{k+1} \neq 0$ so $z_{k+2} = 0$. The $(l+1)$th ray of the admissible chamber is

$$\theta_{1+\frac{l+1}{2}} + \theta_{k+\frac{l+1}{2}} - \theta_{k+1} - \theta_0.$$

This means that there are two nonzero paths $\mathbf{p}$, $\mathbf{q}$ such that either (1) one is from ρ_0 to $\rho_{k+\frac{l+1}{2}}$ and the other one is from ρ_{k+1} to $\rho_{1+\frac{l+1}{2}}$, or (2) one is from ρ_{k+1} to $\rho_{k+\frac{l+1}{2}}$ and the other one is from ρ_0 to $\rho_{1+\frac{l+1}{2}}$. One can show that (2) cannot happen as follows: suppose $\mathbf{p}$ is the nonzero path from ρ_0 to $\rho_{1+\frac{l+1}{2}}$, which is induced by a monomial $x^\alpha y^\beta z^\gamma$ with $\alpha = 0$ due to $x_0 = 0$; if $\mathbf{p}$ is induced by y^β, then it contradicts the assumption that $y_{\frac{l+1}{2}} = 0$; if $\mathbf{p}$ is induced by z^γ, then it contradicts the fact that $z_{k+2} = 0$ since $\frac{l+1}{2} < k$. Let $\mathbf{p}$ be the nonzero path from ρ_0 to $\rho_{k+\frac{l+1}{2}}$. Since $x_0 = 0$ and $y_{\frac{l+1}{2}} = 0$, we know that $\mathbf{p}$ is induced by z^γ. One can see that

$$\gamma = 2k + 1 - \left(k + \tfrac{l+1}{2}\right) = k + 1 - \tfrac{l+1}{2}.$$

Let $\mathbf{q}$ be the nonzero path from ρ_{k+1} to $\rho_{1+\frac{l+1}{2}}$, which is induced by a monomial $x^\alpha y^\beta z^\gamma$. Firstly, since $y_{\frac{l+1}{2}} = 0$, we have $\beta = 0$; otherwise, from the following diagram

$$k+1 \xrightarrow{x^\alpha y^{\beta-1} z^\gamma} \tfrac{l+1}{2} \xrightarrow{\quad y \quad} 1 + \tfrac{l+1}{2},$$

it contradicts the assumption that $\mathbf{q}$ is nonzero. Secondly, since $x_1 = 0$, we have $\alpha \leq 1$. If $\alpha = 1$, then it follows that $\gamma = 0$ from the fact that $z_1 = 0$ so any path induced by xz from ρ_{k+1} is zero. Therefore we get that the path $\mathbf{q}$ is induced by z^γ. One can check that if any $x_i \neq 0$, then there exists a defined (undirected) cycle of type $\mathbf{m}$ with $\mathbf{m} \neq \mathbf{1}$. Thus $x_i = 0$ for all i, and therefore $\mathcal{F}$ is the G-constellation corresponding to Γ_{l+1} in the list of Proposition 5.3.2.

Case (B) $2 \leq l \leq 2k-2$ **is even.** In this case, $\omega(l) = k+\frac{l}{2}$ and $\omega(l-1) = \frac{l}{2}$. Thus we have $y_{k+\frac{l}{2}} = 0$ and $y_{k+1} \neq 0$ so $z_{k+2} = 0$. The $(l+1)$th ray of the admissible chamber is

$$\theta_{k+1+\frac{l}{2}} + \theta_{1+\frac{l}{2}} - \theta_{k+1} - \theta_0.$$

This means that there are two nonzero paths $\mathbf{p}$, $\mathbf{q}$ such that either (i) one is from ρ_0 to $\rho_{k+1+\frac{l}{2}}$ and the other one is from ρ_{k+1} to $\rho_{1+\frac{l}{2}}$, or (ii) one is from ρ_{k+1} to $\rho_{k+1+\frac{l}{2}}$ and the other one is from ρ_0 to $\rho_{1+\frac{l}{2}}$. One can show that (ii) cannot happen as follows: suppose $\mathbf{p}$ is the nonzero path from ρ_{k+1} to $\rho_{k+1+\frac{l}{2}}$, which is induced by a monomial $x^\alpha y^\beta z^\gamma$ with $\alpha \leq 1$ due to $x_1 = 0$; if $\alpha = 1$, then $\gamma = 0$ because $z_1 = 0$ so the path induced by xz from ρ_{k+1} is zero; if $\mathbf{p}$ is induced by y^β or xy^β, then it contradicts the assumption that $y_{k+\frac{l}{2}} = 0$; if $\mathbf{p}$ is induced by z^γ, then it contradicts the fact that $z_1 = 0$. Let $\mathbf{p}$ be the nonzero path from ρ_0 to $\rho_{k+1+\frac{l}{2}}$. We know that $\mathbf{p}$ is induced by z^γ as $x_0 = 0$ and $y_{k+\frac{l}{2}} = 0$. One can see that

$$\gamma = 2k + 1 - \left(k + 1 + \tfrac{l}{2}\right) = k - \tfrac{l}{2}.$$

Let $\mathbf{q}$ be the nonzero path from ρ_{k+1} to $\rho_{1+\frac{l}{2}}$, which is induced by a monomial $x^\alpha y^\beta z^\gamma$. Firstly, note that $\alpha \leq 1$ because $x_1 = 0$. For a contradiction, suppose that $\alpha = 1$. Then we have $\gamma = 0$ since $z_1 = 0$ so that any path induced by xz from ρ_{k+1} is zero. Note that $\beta < \frac{l}{2}$ because $y_{k+\frac{l}{2}} = 0$. Thus xy^β cannot induce a nonzero path from ρ_{k+1} to $\rho_{1+\frac{l}{2}}$. Therefore, $\alpha = 0$ and we can see that the path $\mathbf{q}$ is induced by z^γ with $\gamma = k - \frac{l}{2}$. One can check that if any $x_i \neq 0$, then there exists a defined (undirected) cycle of type $\mathbf{m}$ with $\mathbf{m} \neq \mathbf{1}$. Thus $x_i = 0$ for all i, and therefore $\mathcal{F}$ is the G-constellation corresponding to Γ_{l+1} in the list of Proposition 5.3.2.

$2k$th case: Suppose that $y_{\omega(0)}, y_{\omega(1)}, \ldots, y_{\omega(2k-2)} \neq 0$ and $y_{\omega(2k-1)} = 0$. Note that $2k$th ray of the admissible chamber is

$$\sum_{j=0}^{2k-1} \left(\theta_{\omega(j)+1} - \theta_{\omega(j)}\right) = \theta_{2k} - \theta_0,$$

which means that there exists a nonzero path $\mathbf{p}$ from ρ_0 to ρ_{2k} in $\mathcal{F}$. Since $x_0 = 0$ and $y_{\omega(2k-1)} = 0$, the path $\mathbf{p}$ is induced by the monomial z. One can see that $\mathcal{F}$ is the G-constellation corresponding to Γ_{2k} in the list of Proposition 5.3.2.

$(2k+1)$**th case:** Suppose that $y_{\omega(0)}, y_{\omega(1)}, \dots, y_{\omega(2k)} \neq 0$. Then $\mathcal{F}$ is the G-constellation corresponding to Γ_{2k+1} in the list of Proposition 5.3.2.

Case $x_0 \neq 0$.

Since $\mathcal{F}$ is generated by ρ_0 and ρ_{k+1}, if $x_0 \neq 0$, then $\mathcal{F}$ is generated by ρ_0. Assume that $x_0 \neq 0$. Then $\mathcal{F}$ is a torus invariant G-cluster, i.e. $\mathcal{F}$ is given by a monomial ideal I. The monomials which are not in I form a Nakamura G-graph. In Appendix C, we find all Nakamura G-graphs Γ under the assumption $x_0 \neq 0$, i.e $x \in \Gamma$; this also completes the irreducibility of the moduli space $\mathcal{M}_\theta$.

Since $\mathcal{F}$ is generated by ρ_0, for each ρ_i, there exists a nonzero path from ρ_0 to ρ_i. Moreover, we have three simple observations: (i) the path y^{k+1} from ρ_0 to ρ_{k+1} is zero; otherwise there is a nonzero defined cycle of type $\frac{x}{y^{k+1}}$ around ρ_0 as the following diagram:

$$\rho_0 \xrightarrow{x} \rho_{k+1} \xleftarrow{y^{k+1}} \rho_0 :$$

(ii) any path induced by x^2z is zero because x^2z is a G-invariant monomial: (iii) if $y_0 \neq 0$, then x_{k+1} is zero; otherwise there is a nonzero defined cycle of type $\frac{x^2}{y}$ around ρ_0 as the following diagram:

$$\rho_0 \xrightarrow{x} \rho_{k+1} \xrightarrow{x} \rho_1 \xleftarrow{y} \rho_0.$$

1st case: Suppose that $z_0 = 0$. Note that there exist nonzero paths $\mathbf{p}$ from ρ_0 to ρ_{2k} and $\mathbf{q}$ from ρ_0 to ρ_k. We have two possible cases: (A) $y_0 = 0$: (B) $y_0 \neq 0$:

Case (A) $y_0 = 0$. Since $y_0 = z_0 = 0$, the path $\mathbf{q}$ is induced by x^{2k}. Therefore one can see that $\mathcal{F}$ is the G-constellation corresponding to $\Gamma_1^{\bigtriangledown}$ in the list of Proposition 5.3.2.

Case (B) $y_0 \neq 0$. Observe that $x_{k+1} = 0$. Assume that the path $\mathbf{p}$ is induced by $x^\alpha y^\beta z^\gamma$. Since $z_0 = 0$ and $x_{k+1} = 0$, one can see that $\alpha \leq 1$, $\gamma = 0$, and $\beta \leq k$. From considering the weight of monomials, it follows that the only one possible solution is xy^{k-1}. In a similar way, the only one possible solution for $\mathbf{q}$ is y^k. One can see that $\mathcal{F}$ is the G-constellation corresponding to $\Gamma_1^{\triangle}$ in the list of Proposition 5.3.2.

2nd case: Suppose that $z_0 \neq 0$ and $z_{2k} = 0$. Thus the path induced by z^2 from ρ_0 is zero. We have two possible cases: (A) $y_0 = 0$: (B) $y_0 \neq 0$:

Case (A) $y_0 = 0$. Note that there exists a nonzero path $\mathbf{q}$ from ρ_0 to ρ_k. Assume that the nonzero path $\mathbf{q}$ is induced by $x^\alpha y^\beta z^\gamma$. Note that $\beta = 0$ because $y_0 = 0$. From the fact that any path induced by z^2 or x^2z is zero, it follows that $\gamma \leq 1$ and that the only one possible solution is x^{2k-1}. Therefore $\mathcal{F}$ is the G-constellation corresponding to $\Gamma_2^{\bigtriangledown}$ in the list of Proposition 5.3.2.

Case (B) $y_0 \neq 0$. Observe that $x_{k+1} = 0$ and that there exists a nonzero path $\mathbf{p}$ from ρ_0 to ρ_{2k-1}. Assume that the path $\mathbf{p}$ is induced by $x^\alpha y^\beta z^\gamma$. Since $\alpha \leq 1$ and $\gamma \leq 1$, the only solution is xy^{k-2}. Note that the $(2k-1)$th ray of the admissible chamber is

$$\theta_{k+1+\frac{2k-2}{2}} + \theta_{1+\frac{2k-2}{2}} - \theta_{k+1} - \theta_0 = \theta_{2k} + \theta_k - \theta_{k+1} - \theta_0,$$

which means that there is a nonzero path from ρ_{k+1} to ρ_k or ρ_{2k}. One can show that there are no nonzero paths from ρ_{k+1} to ρ_{2k} as follows: otherwise, since we have a nonzero path induced by z from ρ_0 to ρ_{2k}, we have a nonzero defined cycle around ρ_0 as the following diagram:

$$\rho_0 \xrightarrow{\ x\ } \rho_{k+1} \xrightarrow{\text{nonzero}} \rho_{2k} \xleftarrow{\ z\ } \rho_0.$$

Thus we have a nonzero path $\mathbf{q}$ induced by $x^{\alpha'} y^{\beta'} z^{\gamma'}$ from ρ_{k+1} to ρ_k. Since $x_{k+1} = 0$ and $\beta' \leq k$, it follows that $\alpha' = 0$ so the only possible solution is z. For a nonzero path from ρ_0 to ρ_{k-1}, one can see that the path is induced by y^{k-1}. Therefore $\mathcal{F}$ is the G-constellation corresponding to $\Gamma_2^{\bigtriangleup}$ in the list of Proposition 5.3.2.

$(l+1)$**th case:** Suppose that $z_0, z_{2k}, \ldots, z_{2k+2-l} \neq 0$ and $z_{2k+1-l} = 0$ for $2 \leq l \leq k-1$. Note that the $(2k+1-2l)$th ray of the admissible chamber is

$$\theta_{k+1+\frac{2k-2l}{2}} + \theta_{1+\frac{2k-2l}{2}} - \theta_{k+1} - \theta_0 = \theta_{2k+1-l} + \theta_{k+1-l} - \theta_{k+1} - \theta_0,$$

which means that there is a nonzero path from ρ_{k+1} to ρ_{k+1-l} or ρ_{2k+1-l}. One can show that there are no nonzero paths from ρ_{k+1} to ρ_{2k+1-l} as follows: since we have a nonzero path induced by z^l from ρ_0 to ρ_{2k+1-l}, if so, we have a nonzero defined cycle around ρ_0 as the following diagram:

$$\rho_0 \xrightarrow{\ x\ } \rho_{k+1} \xrightarrow{\text{nonzero}} \rho_{2k+1-l} \xleftarrow{\ z^l\ } \rho_0.$$

Let $\mathbf{q}$ be a nonzero path from ρ_{k+1} to ρ_{k+1-l} induced by $x^{\alpha'} y^{\beta'} z^{\gamma'}$. It follows that the only possible solution is z^l from the fact that x^2z is G-invariant and that $\beta' \leq k$.

Let $\mathbf{p}$ be a nonzero path from ρ_0 to ρ_{2k-l} induced by $x^\alpha y^\beta z^\gamma$. We have two possible cases: (A) $y_0 = 0$: (B) $y_0 \neq 0$:

Case (A) $y_0 = 0$**.** Since $y_0 = 0$, we have $\beta = 0$ and hence the only possible solution is x^{2k-2l} by the fact that x^2z is G-invariant. One can see that $\mathcal{F}$ is the G-constellation corresponding to $\Gamma^{\bigtriangledown}_{l+1}$ in the list of Proposition 5.3.2.

Case (B) $y_0 \neq 0$**.** Note that in this case $x_{k+1} = 0$ so $\alpha \leq 1$. Since $\gamma \leq l$, the only solution is xy^{k-l-1}. For a nonzero path from ρ_0 to ρ_{k-l}, one can see that the path is induced by y^{k-l}. Therefore $\mathcal{F}$ is the G-constellation corresponding to $\Gamma^{\bigtriangleup}_{l+1}$ in the list of Proposition 5.3.2.

Throughout this section, we have proved the following theorem.

Theorem 5.3.11. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{2k+1}(k+1, 1, 2k)$ and θ a generic parameter in the admissible chamber $\mathfrak{C}_a$. Then we have at most $2k+1$ torus invariant G-constellations. Therefore, the moduli space $\mathcal{M}_\theta$ of θ-stable G-constellations is irreducible and isomorphic to the economic resolution of $X = \mathbb{C}^3/G$.*

Proof. By Section 5.3.1, we already know that there exist $2k+1$ θ-stable G-iraffes. Thus we have at least $2k+1$ torus invariant G-constellations lying over the birational component. From Remark 4.1.9, it follows that $Y_\theta = \mathcal{M}_\theta$. □

5.4 Irreducibility for type $\frac{1}{12}(1, 7, 5)$

In this section, we show that for the group of type $\frac{1}{12}(1, 7, 5) \sim \frac{1}{12}(7, 1, 11)$, the moduli space $\mathcal{M}_\theta$ is irreducible by finding all θ-stable torus invariant G-constellations for GIT parameter $\theta \in \mathfrak{C}_a$. As is in the previous section, we use $\frac{1}{r}(b, 1, r-1)$ notation.

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the finite subgroup of type $\frac{1}{12}(7, 1, 11)$. One can see that G-invariants monomials are generated by

$$x^{12}, x^5y, x^3y^3, xy^5, y^{12}, x^7z, x^2z^2, xz^7, z^{12}.$$

Table 5.4.1 presents the monomials of weight i.

The list of G-iraffes is in Table 4.5.1. In this section, we prove that θ-stable torus invariant G-constellations are all induced by G-iraffes.

We recall the admissible Weyl chamber for the group of type $\frac{1}{12}(7, 1, 11)$ (see

Weight	Monomials
1	$x^7, y, x^2z, xz^6, z^{11}$
2	x^2, y^2, xz^5, z^{10}
3	$x^9, x^2y, y^3, x^4z, xz^4, z^9$
4	$x^4, x^2y^2, y^4, xz^3, z^8$
5	$x^{11}, x^4y, x^2y^3, y^5, x^6z, xz^2, z^7$
6	$x^6, x^4y^2, x^2y^4, y^6, xz, z^6$
7	x, y^7, z^5
8	x^8, xy, y^8, x^3z, z^4
9	x^3, xy^2, y^9, z^3
10	$x^{10}, x^3y, xy^3, y^{10}, x^5z, z^2$
11	$x^5, x^3y^2, xy^4, y^{11}, z$

Table 5.4.1: Monomials of weight i for $G = \frac{1}{12}(7, 1, 11)$

Section 4.5.2). The admissible set of simple roots is

$$\Delta_a = \begin{Bmatrix} \varepsilon_{11} - \varepsilon_6,\ \varepsilon_6 - \varepsilon_{10},\ \varepsilon_{10} - \varepsilon_5,\ \varepsilon_5 - \varepsilon_8,\ \varepsilon_8 - \varepsilon_3,\ \varepsilon_3 - \varepsilon_1, \\ \varepsilon_1 - \varepsilon_9,\ \varepsilon_9 - \varepsilon_4,\ \varepsilon_8 - \varepsilon_7,\ \varepsilon_7 - \varepsilon_2,\ \varepsilon_2 - \varepsilon_0 \end{Bmatrix},$$

and the corresponding permutation ω is

$$\omega = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0 & 2 & 7 & 4 & 9 & 1 & 3 & 8 & 5 & 10 & 6 & 11 \end{pmatrix}.$$

From this, the rays of the admissible chamber $\mathfrak{C}_a$ are the row vectors of the following matrix:

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.4.1}$$

Let $\mathcal{F}$ be a θ-stable torus invariant G-constellation. Recall that for a genuine monomial $\mathbf{m}$, $\mathbf{m}_{(i)}$ denotes the linear map corresponding to the path from ρ_i induced by $\mathbf{m}$.

We have two cases: (I) $x_0 = 0$: (II) $x_0 \neq 0$.

Case (I) $x_0 = 0$.

Let l be the smallest integer such that the linear map $y_{\omega(l)}$ is zero. For each l, the torus invariant G-constellation $\mathcal{F}$ corresponds to the G-iraffe of σ_{l+1} in Table 4.5.1.

As an example, we consider the case where $l = 0, 3, 7, 11$. For the other cases, one can show the assertion by considering $(l+1)$th row vector of the matrix (5.4.1) in a similar manner.

Case $l = 0$. This means that $y_0 = 0$. Since the first row of the matrix is $\theta_1 - \theta_0$, there exists a nonzero path from ρ_0 to ρ_1; otherwise, the submodule of $\mathcal{F}$ generated by ρ_0 is negative with respect to the first row. From the assumption that $x_0 = y_0 = 0$, the path is induced by z^{11} because the x,y-actions at ρ_0 are zero. One can see that $\mathcal{F}$ is given by the G-iraffe corresponding to σ_1 in Table 4.5.1.

Case $l = 3$. This means that $y_0, y_2, y_7 \neq 0$ and that $y_4 = 0$. Since the fourth row of the matrix (5.4.1) is

$$\theta_8 + \theta_5 + \theta_3 + \theta_1 - \theta_7 - \theta_4 - \theta_2 - \theta_0,$$

there exists a nonzero path $\mathbf{p}$ from ρ_4 to one of ρ_1, ρ_3, ρ_5 or ρ_8; otherwise, $\mathcal{F}$ has a submodule which is negative with respect to the vector above. Firstly, note that $x_1 = 0$ by $y_0x_1 = x_0y_7$ and that $z_1 = z_3 = 0$, by $y_0, y_2 \neq 0$; otherwise, $\mathcal{F}$ has a nonzero cycle around ρ_2 induced by yz. From this, it follows that any paths from ρ_4 induced by y, x^4, x^3z, z^2 are zero: indeed, one can see that an arrow of each path is zero;

$$y_4 = x_4x_{11}x_6x_1 = x_4x_{11}x_6z_1 = z_4z_3 = 0.$$

From Table 5.4.1, the nonzero path $\mathbf{p}$ is induced by x^3 or z.

For a contradiction, suppose that $z_4 = 0$. Then $\mathbf{p}$ is induced by x^3, so $x_4x_{11}x_6$ is nonzero. Thus z_0 is zero; otherwise, it contradicts that $z_0x_{11} = x_0z_7 = 0$. Then nonzero paths from ρ_0 to ρ_5 or ρ_8 cannot exist. Thus there exists a nonzero path from ρ_2 to ρ_5 or ρ_8. By considering all possible monomials of suitable weights, it contradicts the fact that any paths from ρ_2 induced by x^6, x^2y, y^3, xz, z^2 are zero.

Considering the row vector above, we know that there exists a nonzero path from ρ_2 to ρ_1, which is induced by x^5 or z. For a contradiction, suppose that the path is induced by x^5. Since $x_2x_9x_4x_{11}x_6$ is nonzero, it follows that $x_2x_9z_4 = z_2x_1x_8$ is nonzero, which contradicts that $x_1 = 0$. Thus z_2 is nonzero and x^5 is zero.

Consider the vertex ρ_7. We have a nonzero path $\mathbf{q}$ from ρ_7 to ρ_5 because there are no nonzero paths from ρ_0 to ρ_5. Note that any paths from ρ_7 induced by x^3y, y^{10}, z^5, x^7 are zero. Moreover, paths induced by x^2z are zero; otherwise, we have the following nontrivial undirected cycle around ρ_7:

$$\rho_7 \xrightarrow{y} \rho_8 \xleftarrow{x^z} \rho_7.$$

Thus $\mathbf{q}$ is induced by z^2.

Lastly, one can see that we have a nonzero path from ρ_0 to ρ_8 induced by z^4. Therefore, $\mathcal{F}$ is given by the G-iraffe corresponding to σ_4 in Table 4.5.1.

Case $l = 7$. Thus we have that $y_0, y_2, y_7, y_4, y_9, y_1, y_3 \neq 0$ and that $y_8 = 0$. Then $x_0 = x_1 = x_2 = x_3 = x_4 = 0$. Consider the eighth row vector of the matrix (5.4.1):

$$\theta_{10} + \theta_5 - \theta_7 - \theta_0.$$

Then there exists a nonzero path $\mathbf{p}$ from ρ_7 to one of ρ_5 or ρ_{10}. As any paths from ρ_7 induced by x^2, y^2, z^3 are zero, one can see that $\mathbf{p}$ is induced by z^2. Moreover, we have the nonzero path induced by z^2 from ρ_0 to ρ_{10}. Therefore, $\mathcal{F}$ is given by the G-iraffe corresponding to σ_8 in Table 4.5.1.

Case $l = 11$. As by the assumption we have that y^{11} induces a nonzero path from ρ_0 to ρ_{11}, $\mathcal{F}$ is given by the G-iraffe corresponding to σ_{12} in Table 4.5.1.

Case (II) $x_0 \neq 0$.

As $x_0 \neq 0$, the paths induced by y^7, z^5 from ρ_0 to ρ_7 are zero. Considering the first and the last row vectors of the matrix (5.4.1), we know that there exist a nonzero path from ρ_0 to ρ_1 and a nonzero path from ρ_0 to ρ_{11}. The former can be induced by x^7, y, x^2z and the latter can be induced by x^5, x^3y^2, xy^4, z. We have the following five cases:

(1) $x_0 \neq 0$ and $x_7 = 0$.

(2) $x_0, x_7 \neq 0$ and $x_2 = 0$.

(3) $x_0, x_7, x_2 \neq 0$ and $x_6 = 0$.

(4) $x_0, x_7, x_2, x_6 \neq 0$ and $x_1 = 0$.

(5) $x_0, x_7, x_2, x_6, x_1 \neq 0$.

Case (1) $x_0 \neq 0$ and $x_7 = 0$.

Since the path $x^2_{(0)}$ induced by x^2 from ρ_0 is x_0x_7, it is zero. As we have a nonzero path ρ_0 to ρ_1, y_0 is nonzero.

We have the following five cases (1-A)-(1-E):

(1-A) $y_2 = 0$.

(1-B) $y_2 \neq 0$ and $y_4 = 0$.

(1-C) $y_2, y_4 \neq 0$ and $y_3 = 0$.

(1-D) $y_2, y_4, y_3 \neq 0$ and $y_5 = 0$.

(1-E) $y_2, y_4, y_3, y_5 \neq 0$ and $y_6 = 0$.

All of (1-A),(1-B),(1-C),(1-D),and (1-E) give G-constellations corresponding to the G-iraffes corresponding to τ_2, τ_3, τ_6, τ_7, and τ_9, respectively.

As examples, we investigate Case (1-B) and Case (1-D).

Case (1-B) $y_2 \neq 0$ and $y_4 = 0$. By the assumption, we have that any paths from ρ_0 induced by x^2, y^5, z^5 are zero. Note that any paths from ρ_0 induced by xy^4 are zero; otherwise, the submodule generated by ρ_4 is supported on ρ_4, ρ_{10} as $y_4, z_4, y_{10}, z_{10}, x_{10}$ are zero, so it is negative with respect to the fourth row of the matrix (5.4.1). Considering the eleventh row vector, we know that z_0 is nonzero. Furthermore, as $\theta(\rho_4)$ is negative, the path induced by y^4 from ρ_0 is zero.

Considering the eighth row vector $\theta_5 + \theta_{10} - \theta_7 - \theta_0$, we have a nonzero path from ρ_7 to ρ_{10} or ρ_5. The monomials that can induce a path from ρ_7 to ρ_{10} are

$$x^9, x^2y, y^3, x^4z, xz^4, z^9,$$

which induce zero paths at ρ_7. The only possible solutions are the nonzero paths from ρ_7 induced by z^2, y^3. If $z^2_{(7)}$ is zero, then we have a nonzero path from ρ_0 to ρ_5. This implies that $xz^2_{(0)}$ is nonzero, which contradicts to that $z^2_{(7)}$ is zero. Thus $z^2_{(7)}$ is nonzero, and z_7, z_6, z_{11} are nonzero. From this, we know that the path $xy^3_{(0)}$ is zero because xy^3 is of the same weight as z^2, which induces nonzero path from ρ_0.

Consider the third row vector $\theta_1 + \theta_3 + \theta_8 - \theta_7 - \theta_2 - \theta_0$. Suppose that y_7 is zero. Then it follows that x_1 is zero and that there exist no nonzero paths from ρ_0 or

ρ_7 to ρ_8. Thus the nonzero path induced by z^4 from ρ_7 is nonzero. As x_7 is zero, x_3 is zero. Moreover, we have a nonzero path from ρ_2 to ρ_8, which can be induced by $x^6, x^4y^2, x^2y^4, y^6, xz, z^6$. Note that paths induced by $x^6, x^4y^2, x^2y^4, y^6, xz, z^6$ from ρ_2 are zero because:

$$x_2x_9x_4x_{11}x_6x_1 = y_2x_3 = y_2y_3y_4 = z_2x_1 = z_2z_1 = 0,$$

which is a contradiction. Thus y_7 is nonzero. Furthermore, as $\theta(\rho_9)$ is negative, the path $xy^2_{(0)}$ is zero because $x^2y^2_{(0)}$ and $xy^3_{(0)}$ are zero. In addition, from the fact that $\theta(\rho_4), \theta(\rho_9)$ are negative, it follows that $z^3_{(0)}$ is zero.

Note that z_4 is nonzero; otherwise, there are no nonzero arrows from ρ_4 so the vertex simple $\mathbb{C}\rho_4$ is a submodule of $\mathcal{F}$ with $\theta(\rho_4) < 0$. As z_4, x_{10} are nonzero and x_8, y_4 are zero, we have x_9, y_9 are zero, and hence $z_9 \neq 0$ because $\theta(\rho_9) < 0$. Similarly, as y_9 is zero, from that

$$y_2x_3 = x_2y_9, \quad z_4x_3 = x_4z_{11},$$

we have $x_3 = x_4 = 0$. We know that z_2 or x_2 is nonzero, by considering the fourth row vector $\theta_1 + \theta_3 + \theta_5 + \theta_8 - \theta_7 - \theta_4 - \theta_2 - \theta_0$. As z_9, x_1 are nonzero, it follows that both z_2, x_2 are nonzero from the fact that $x_2z_9 = z_2x_1$.

One can see that $\mathcal{F}$ corresponds to the G-iraffe of τ_3 in Table 4.5.1.

Case (1-D) $y_2, y_4, y_3 \neq 0$ **and** $y_5 = 0$**.** As $x_0 \neq 0, y_5 = 0, x_7 = 0$, the paths $x^2_{(0)}, y^5_{(0)}, z^5_{(0)}$ from ρ_0 are zero. Considering the tenth row vector $\theta_6 + \theta_{11} - \theta_7 - \theta_0$ of the matrix (5.4.1), we know that there exists a nonzero path from one of ρ_0 or ρ_7 to ρ_6. In both cases, $\mathcal{F}$ contains a nonzero path from one of ρ_0 to ρ_6 as x_0 is nonzero. The monomials which can induce the nonzero path are $x^6, x^4y^2, x^2y^4, y^6, xz, z^6$. Note that only xz can induce a nonzero path as the paths $x^2_{(0)}, y^5_{(0)}, z^5_{(0)}$ are zero. In addition, one can see that x_6 is zero; otherwise, $x_0x_7z_7 = x_0z_7x_6$ is nonzero, which contradicts $x_7 = 0$.

Consider the sixth row vector of the matrix (5.4.1):

$$\theta_3 + \theta_5 + \theta_8 + \theta_{10} - \theta_9 - \theta_7 - \theta_4 - \theta_0.$$

Thus $\mathcal{F}$ has a nonzero path coming to ρ_3 from one of $\rho_0, \rho_4, \rho_7, \rho_9$. Considering all possible monomials at each vertex, one can see that we have only one possible nonzero path induced by y^3 from ρ_0; for example, $x^5, x^3y^2, xy^4, y^{11}, z$ can induce a path from ρ_4 to ρ_3; the paths from ρ_4 induced by x^5, y^2, z are zero because x_6, y_5, z_4

are zero. From the fact that y_3, y_4 are nonzero, it follows that the path $y^5_{(0)}$ is nonzero.

We now show $y^3_{(7)}$ is nonzero. Note that there are no nonzero paths ρ_7 to ρ_5; otherwise, we have the following nonzero cycle:

$$\rho_0 \xrightarrow{x} \rho_7 \xrightarrow{\text{nonzero}} \rho_5 \xleftarrow{y^5} \rho_0.$$

Consider the eighth row vector of the matrix (5.4.1):

$$\theta_{10} + \theta_5 - \theta_7 - \theta_0.$$

Thus $\mathcal{F}$ contains a nonzero path from ρ_7 to ρ_{10}. The monomials which can induce the path are the monomials of weight 3:

$$x^9, x^2y, y^3, x^4z, xz^4, z^9.$$

Note that as xz^2 is of the same weight as y^5, the fact that $y^5_{(0)}$ is nonzero implies that $xz^2_{(0)}$ is zero. Thus the nonzero path from ρ_7 to ρ_{10} is induced by y^3. One can see that $\mathcal{F}$ is the torus invariant G-constellation associated with the G-iraffe of τ_7 in Table 4.5.1.

Case (2) $x_0, x_7 \neq 0$ **and** $x_2 = 0$**.**

We have the following two cases: (2-A) $y_0 = 0$: (2-B) $y_0 \neq 0$. These cases (2-A) and (2-B) give G-constellations corresponding to τ_1 and τ_5 in Table 4.5.1, respectively. Here we show that there is a G-iraffe giving $\mathcal{F}$ for Case (2-B).

Case (2-B) $y_0 \neq 0$**.** As $x_0^2 = x_0x_7$ is nonzero, $y_1 = 0$. One can see that the monomials x^3, y^2, x^2z, z^5 induce zero paths from ρ_0 and that there are no nonzero paths from ρ_2 to ρ_1: otherwise we have a nonzero cycle. Firstly, z_0 is nonzero by the eleventh row vector of the matrix (5.4.1). Secondly, considering the 3rd row vector

$$\theta_1 + \theta_+\theta_8 - \theta_7 - \theta_2 - \theta_0$$

of the matrix (5.4.1), we can see that there exists a nonzero path from ρ_2 to one of ρ_1, ρ_3,ρ_8. One can show that any paths from ρ_2 induced by monomials of weight 6 are zero because $x_2 = z_2 = y_2y_3 = 0$. Thus we have a nonzero path from ρ_2 to ρ_3. The monomial y only can induce a nonzero path from ρ_2 to ρ_3. From this, we have that x_1, y_7, x_8, y_2 are nonzero and that x_3 is zero, so the paths $z^4_{(0)}$ and $xz^4_{(0)}$ are zero. From this, by the negativeness of $\theta(\rho_4)$ and $\theta(\rho_9)$, we can see that the paths $z^3_{(0)}$ and $xz^3_{(0)}$ are zero. Indeed, if $xz^3_{(0)}$ is nonzero, then we can see that $\mathbb{C}\rho_4$ is a submodule of $\mathcal{F}$. By a similar reason, we can see that the paths $z^3_{(0)}$ is zero.

Consider the eighth row vector of the matrix (5.4.1):

$$\theta_{10} + \theta_5 - \theta_7 - \theta_0.$$

There exists a nonzero path going to ρ_5 from one of ρ_0, ρ_7. In both cases, we have a nonzero path from ρ_0 to ρ_7 as x_0 is a nonzero arrow from ρ_0 to ρ_7. Among the monomials of weight 5, the monomial xz^2 only can induce a nonzero path from ρ_0. In particular, z_{11}, x_{10} are nonzero.

As $x_4z_{11} = z_4x_3 = 0$, we have that $x_4 = 0$. Since $\theta(\rho_9)$ is negative, at least one of x_9, y_9, z_9 is nonzero, which implies that x_9 is nonzero; if $x_9 = 0$, then $y_9 = z_9 = 0$ because $x_9z_4 = z_9x_8$ and $x_9y_4 = y_9x_{10}$.

Let us consider the fifth row vector

$$\theta_1 + \theta_3 + \theta_5 + \theta_8 + \theta_{10} - \theta_9 - \theta_7 - \theta_4 - \theta_2 - \theta_0.$$

As there exist no nonzero paths from ρ_2 to any of $\rho_1, \rho_5, \rho_8, \rho_{10}$, there exists a nonzero path $\mathbf{p}$ from ρ_4 to one of $\rho_1, \rho_5, \rho_8, \rho_{10}$. Note that $z^2_{(4)}, y^2_{(4)}$ are zero as $z_3 = 0$ and $y_5 = 0$, respectively. The following monomials can induce $\mathbf{p}$:

$$x^3, xy^2, y^9, z^3, x^7, y, x^2z, xz^6, z^{11}, x^4, x^2y^2, y^4, xz^3, z^8, x^6, x^4y^2, x^2y^4, y^6, xz, z^6.$$

The monomial y only can induce the nonzero path $\mathbf{p}$, so y_4 is nonzero. From the fact that $x_9y_4 = y_9x_{10}$, it follows that y_9 is nonzero. In addition, if z_4 is nonzero, then we have the following nonzero cycle:

$$\rho_4 \xrightarrow{z} \rho_3 \xleftarrow{xy} \rho_7 \xrightarrow{z^2} \rho_5 \xleftarrow{y} \rho_4.$$

Case (3) $x_0, x_7, x_2 \neq 0$ and $x_6 = 0$.

We have the following two cases: (3-A) $y_0 = 0$: (3-B) $y_0 \neq 0$. In these cases, in a similar way to Case (2), it can be proved that (3-A) and (3-B) give G-constellations corresponding to τ_4 and τ_8 in Table 4.5.1, respectively.

Case (4) $x_0, x_7, x_2, x_6 \neq 0$ and $x_1 = 0$.

In a similar manner to Case (2), we can show that this case corresponds to the cone τ_{10} in Table 4.5.1.

Case (5) $x_0, x_7, x_2, x_6, x_1 \neq 0$**.**

In a similar way as above, one can show that this case corresponds to the cone τ_0 in Table 4.5.1.

Conclusion.

We have seen that for the finite group G of type $\frac{1}{12}(7,1,11)$ and a parameter θ in the admissible chamber $\mathfrak{C}$, there exist exactly 23 θ-stable torus invariant G-constellations. By Remark 4.1.9, we have shown that $\mathcal{M}_\theta$ is irreducible, so $\mathcal{M}_\theta$ is isomorphic to the economic resolution of $\mathbb{C}^3/G$.

Appendix A

$\mathcal{M}_0$ is irreducible for $G = \frac{1}{r}(1, a, r-a)$

Let G be the finite subgroup of $\mathrm{GL}_3(\mathbb{C})$ of type $\frac{1}{r}(1, a, r-a)$ and X the quotient space $\mathbb{C}^3/G$. Consider the moduli space $\mathcal{M}_0$ of 0-semistable G-constellations for $0 = (0, \ldots, 0)$. By definition,

$$\mathcal{M}_0 = \operatorname{Spec} \mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$$

parametrises 0-semistable G-constellations up to S-equivalence. Note that every G-constellation is 0-semistable.

Recall that $\mathbf{m}_{(i)}$ denotes the linear map induced by the action of a genuine monomial $\mathbf{m} \in \overline{M}_{\geq 0}$ on the vector space $\mathbb{C}\rho_i$.

Proposition A.0.1. *Let $\mathcal{F}$ be a G-constellation for the finite group G of type $\frac{1}{r}(1, a, r-a)$. We have the following:*

(i) $\mathcal{F}$ is 0-stable if and only if it is isomorphic to $\mathcal{O}_Z$ for a free G-orbit Z in $\mathbb{C}^3$.

(ii) if $\mathcal{F}$ is not 0-stable, then $\mathcal{F}$ is S-equivalent to $\bigoplus_\rho \mathcal{O}_0 \otimes \rho$, where $\mathcal{O}_0$ is the skyscraper sheaf at the origin $(0,0,0)$. Therefore all strictly 0-semistable G-constellations collapse to a point in the moduli space.

Moreover, the moduli space $\mathcal{M}_0$ is isomorphic to $X = \mathbb{C}^3/G$.

Proof. If $\mathcal{F}$ is 0-stable, then $\mathcal{F}$ has no nonzero proper submodules, which means that $\mathcal{F}$ is simple. Let e_ρ be a basis of $\mathbb{C}\rho$. Then the submodule generated by e_ρ is equal to $\mathcal{F}$. This means that there exists a nonzero path from ρ to ρ' for any other ρ'.

From this, if $\mathcal{F}$ is 0-stable, it follows that there exists a nonzero cycle passing through every vertex. Then $\mathcal{F}$ is supported on a free G-orbit Z in $\mathbb{C}^3$, and hence $\mathcal{F}$ is isomorphic to $\mathcal{O}_Z$ by Lemma 2.2.12. This proves (i).

For (ii), assume that $\mathcal{F}$ is not 0-stable so there are no nonzero cycles passing through all vertices; otherwise, there are no nonzero proper submodules, which implies that $\mathcal{F}$ is 0-stable. Firstly, note that $\mathcal{F}$ should be supported on the origin as G acts freely outside of the origin.

We claim that there are no nonzero cycles; suppose that there is a nonzero cycle around ρ_0 and write the nontrivial monomial $\mathbf{m} = x^{m_1}y^{m_2}z^{m_3}$ corresponding to the cycle, so that $\mathbf{m}_{(0)}$ is nonzero. Assume that $m_1 \geq 1$ so that the cycle must pass through the vertex $\rho_1 = \mathrm{wt}(x)$. Since

$$\mathbf{m}_{(0)} = x_0 \cdot \left(\frac{\mathbf{m}}{x}\right)_{(1)}$$

by the commutation relations, the linear map $\left(\frac{\mathbf{m}}{x}\right)_{(1)}$ induced by the monomial $\frac{\mathbf{m}}{x}$ at ρ_1 is nonzero. Thus the linear map induced by $\mathbf{m}$ at ρ_1

$$\mathbf{m}_{(1)} = \left(\frac{\mathbf{m}}{x}\right)_{(1)} \cdot x_0$$

is nonzero. Thus we know that there exists a nonzero path from ρ_0 to ρ_1 and that $\mathbf{m}_{(1)}$ is nonzero. Since 1 is coprime to r, we can get a nonzero cycle induced by x^r which is nonzero. For the other cases, e.g. $m_2 \geq 1$, we can find a nonzero cycle similarly.

Since $\mathcal{F}$ contains no nonzero cycles, there exists a vertex ρ_k such that the linear map induced by any nontrivial path to ρ_k is zero. Write $\mathcal{F}_1 = \oplus_{i \neq k} V_i$, which is a submodule of $\mathcal{F}$. Then $\mathcal{F}/\mathcal{F}_1$ is a vertex simple and is isomorphic to $\mathcal{O}_0 \otimes \rho_k$. Since $\mathcal{F}_1$ does not have nonzero cycles, we can deduce that $\mathcal{F}$ is S-equivalent to $\bigoplus_\rho \mathcal{O}_0 \otimes \rho$.

To prove (iii), firstly note that from the classical invariant theory, the set of cycles induced by genuine monomials

$$\left\{\mathbf{m}_{(i)} \,\middle|\, \mathbf{m} \in \overline{M}_{\geq 0}, i \in I\right\}$$

generates the coordinate ring $\mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$ of $\mathcal{M}_0$.

We define an algebra homomorphism ψ from $\mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$ to $\mathbb{C}[x,y,z]^G$ by

$$\psi\colon \mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)} \to \mathbb{C}[x,y,z]^G, \quad \mathbf{m}_{(0)} \mapsto \mathbf{m}.$$

The algebra homomorphism ψ is clearly surjective. To prove the injectivity, it suffices to show that $\mathbf{m}_{(i)} = \mathbf{m}_{(0)}$ in $\mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)}$ for all $i \in I$ if $\mathbf{m} \in \overline{M}_{\geq 0}$. Assume that $\mathbf{m} = (m_1, m_2, m_3) \in \overline{M}_{\geq 0}$ with $m_1 \geq 1$. Then the cycle must pass the vertex $\rho_1 = \mathrm{wt}(x)$. Since

$$\mathbf{m}_{(0)} = x_0 \cdot \left(\frac{\mathbf{m}}{x}\right)_{(1)} = \left(\frac{\mathbf{m}}{x}\right)_{(1)} \cdot x_0 = \mathbf{m}_{(1)},$$

it is proved that $\mathbf{m}_{(i)} = \mathbf{m}_{(0)}$ for all $i \in I$ by the fact that 1 is coprime to r. For the other cases (e.g. $m_2 \geq 1$), we can prove the assertion similarly as a is coprime to r. □

Remark A.0.2. In the proof of the proposition above, we also proved that the quotient variety $\mathbb{C}^3/G$ can be embedded into $\mathcal{M}_0$ as a closed subvariety for any finite abelian subgroup $G \subset \mathrm{GL}_3(\mathbb{C})$, because there exists an algebra homomorphism

$$\mathbb{C}[\operatorname{Rep} G]^{\mathrm{GL}(\delta)} \to \mathbb{C}[x, y, z]^G,$$

which is surjective. ♦

Appendix B

Example: G-graphs which are not G-iraffes

In [25] Nakamura assumed that $U(\Gamma)$ has a torus fixed point for any Nakamura G-graph Γ i.e. every G-graph in his sense is a G-iraffe. His assumption implies that every torus invariant G-cluster lies over the birational component of G-Hilb. However, Craw, Maclagan and Thomas [5] showed that there exists a torus invariant G-cluster which is not over the birational component.

Example B.0.1 (Craw, Maclagan and Thomas [5])**.** Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{14}(1,9,11)$. Note that G is isomorphic to $\frac{1}{7}(1,2,4) \times \frac{1}{2}(1,1,1)$. Consider the monomial ideal

$$I = \langle y^2z, xz^2, xy^2, x^2y, yz^2, x^2z, x^4, y^4, z^4 \rangle$$

and the corresponding Nakamura G-graph

$$\Gamma = \{1, x, x^2, x^3, y, y^2, y^3, z, z^2, z^3, xy, xz, yz, xyz\}.$$

Craw, Maclagan and Thomas [5] showed that this ideal does not lie over the birational component using Gröbner basis techniques.

We show this by proving the G-graph Γ is not a G-iraffe. One can calculate the semigroup $S(\Gamma)$ and notice that $S(\Gamma)$ is generated as a subsemigroup in M by $\frac{xy^2}{z^3}, \frac{yz^2}{x^3}, \frac{x^2z}{y^3}, \frac{y^2z}{x}$. Note that

$$\frac{xy^2}{z^3} \cdot \frac{yz^2}{x^3} \cdot \frac{x^2z}{y^3} = 1$$

and hence $\frac{xy^2}{z^3} \in S(\Gamma) \cap (S(\Gamma))^{-1} \neq \{\mathbf{1}\}$. Thus $U(\Gamma)$ does not have a torus fixed point. Indeed, the cone $\sigma(\Gamma)$ is the cone generated by $\frac{1}{14}(7,7,7)$ so it is not a full dimensional cone. Therefore the G-cluster $C(\Gamma) = \mathbb{C}[x,y,z]/I$ does not lie over the birational component. ◆

Remark B.0.2. Craw, Maclagan, and Thomas [5] provided an equivalent condition using Gröbner basis for a monomial ideal to be over the birational component. In the terms of G-iraffes, the condition is equivalent for a Nakamura G-graph to be a G-iraffe. ◆

Example B.0.3 (Reid)**.** Let $G \subset \mathrm{SL}_4(\mathbb{C})$ be the group of type $\frac{1}{30}(1,6,10,13)$ with coordinates x,y,z,t. Consider the monomial ideal

$$I = \left\langle \begin{matrix} x^6, x^3y, x^3t, x^2z, x^2t^2, xy^2, xyt, xzt, xt^3, \\ y^5, y^4z, y^3t, y^2zt, yz^2, yt^2, z^3, z^2t, zt^2, t^4 \end{matrix} \right\rangle$$

and the corresponding Nakamura G-graph

$$\Gamma = \left\{ \begin{matrix} 1, x, x^2, x^3, x^4, x^5, y, y^2, y^3, y^4, z, z^2, \\ t, t^2, t^3, xy, x^2y, xz, xz^2, xt, x^2t, xt^2, \\ yz, y^2z, y^3z, yt, y^2t, zt, xyz, yzt \end{matrix} \right\}.$$

Note that $\frac{y^2zt}{x^5}, \frac{x^3y}{t^3}, \frac{x^2t^2}{y^3z}$ are in the semigroup $S(\Gamma)$ and

$$\frac{y^2zt}{x^5} \cdot \frac{x^3y}{t^3} \cdot \frac{x^2t^2}{y^3z} = 1.$$

Thus $\frac{y^2zt}{x^5} \in S(\Gamma) \cap (S(\Gamma))^{-1} \neq \{\mathbf{1}\}$. Thus $U(\Gamma)$ does not have a torus fixed point. Therefore the G-cluster $C(\Gamma) = \mathbb{C}[x,y,z,t]/I$ does not lie over the birational component. ◆

Remark B.0.4. Reid used the ideal in Example B.0.3 to provide a case where G-Hilb has a 5-dimensional component even if G is a subgroup of $\mathrm{GL}_4(\mathbb{C})$. ◆

Appendix C

Nakamura G-graphs for type $\frac{1}{2k+1}(k+1,1,2k)$

Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{2k+1}(k+1,1,2k)$ and θ a generic parameter in the admissible chamber $\mathfrak{C}_a$ (see Section 5.3.2).

If $x_0 \neq 0$, then any θ-stable G-constellation is generated by ρ_0, so it is a G-cluster. Thus we have a 1-to-1 correspondence between the set

$$\{\theta\text{-stable torus invariant } G\text{-constellations with } x_0 \neq 0\}$$

and the set

$$\{\text{Nakamura } G\text{-graphs } \Gamma \text{ containing } x\}.$$

In this section, we classify all Nakamura G-graphs containing x. By doing that, we prove that the number of θ-stable torus invariant G-constellations with $x_0 \neq 0$ is $2k$.

Lemma C.0.1. *Let $G \subset \mathrm{GL}_3(\mathbb{C})$ be the group of type $\frac{1}{2k+1}(k+1,1,2k)$ and θ a generic parameter in the admissible chamber $\mathfrak{C}_a$. Assume that Γ is a G-graph containing x. Then Γ has the following properties:*

(i) $y^{k+1}, z^k \notin \Gamma$.

(ii) $yz, x^2z \notin \Gamma$.

(iii) $x^2 \notin \Gamma$, *if* $y \in \Gamma$.

Furthermore, if Γ is θ-stable, then $z^l \in \Gamma$ for $1 \leq l \leq k$ implies that $xz^l \in \Gamma$.

Proof. The assertion (i)-(iii) are straightforward from the definition of G-graphs.

Suppose that Γ is θ-stable and that Γ contains z^l for $1 \leq l \leq k$. Note that by Section 5.3.2 the $(2k+1-2l)$th ray of the admissible chamber is

$$\theta_{2k+1-l} + \theta_{k+1-l} - \theta_{k+1} - \theta_0,$$

which implies that there is a nonzero path from ρ_{k+1} to ρ_{k+1-l} or ρ_{2k+1-l}. Remember that the existence of a nonzero path from ρ_i to ρ_j is equivalent to the condition that $\mathbf{m}_{\rho_i}$ divides $\mathbf{m}_{\rho_j}$ where $\mathbf{m}_{\rho_i}$, $\mathbf{m}_{\rho_j}$ are the corresponding monomials in Γ. Since z^l is of weight $2k+1-l$, z^l is the monomial of weight $2k+1-l$ in Γ and x is the monomial of weight $k+1$ in Γ, there are no nonzero paths from ρ_{k+1} to ρ_{2k+1-l}. Thus there exists a nonzero path from ρ_{k+1} to ρ_{k+1-l}.

Assume $\mathbf{m} = x^\alpha y^\beta z^\gamma$ is a unique monomial of weight $k+1-l$ in Γ. Since x divides $\mathbf{m}$, we get $\alpha \geq 1$. Note that $y^{k+1} \notin \Gamma$ from Lemma C.0.1 implies $\beta \leq k$. Since $x^{\alpha-1}y^\beta z^\gamma$ is a genuine monomial of weight $2k+1-l$, it follows that $\mathbf{m} = xz^l$. Therefore, $\mathbf{m} = xz^l$ is in Γ. □

Proposition C.0.2. *Let G be the group of type $\frac{1}{2k+1}(k+1, 1, 2k)$ and θ a generic parameter in the admissible chamber $\mathfrak{C}_a$. Assume that Γ is a θ-stable G-graph containing x. Then Γ is equal to either $\Gamma_l^\triangle$ or $\Gamma_l^\triangledown$ in the list of Proposition 5.3.2 for some $1 \leq l \leq k$.*

Proof. Let Γ be a θ-stable G-graph containing x. From Lemma C.0.1, there exists l with $1 \leq l \leq k$ such that $1, z, z^2, \ldots, z^{l-1} \in \Gamma$ and $z^l \notin \Gamma$. The G-graph Γ contains the monomials $x, xz, xz^2, \ldots, xz^{l-1}$ and $xz^l \notin \Gamma$ by Lemma C.0.1.

We have two cases: (A) $y \in \Gamma$: (B) $y \notin \Gamma$:

Case (A) $y \in \Gamma$**.** Since Γ has $2k+1$ monomials and $x^2z \notin \Gamma$, Γ is $\Gamma_l^\triangledown$, i.e.

$$\Gamma = \left\{ \begin{array}{cccccccc} 1 & x & x^2 & x^3 & \ldots & x^{2k-2l} & x^{2k-2l+1} & x^{2k-2l+2} \\ z & xz & & & & & & \\ \ldots & \ldots & & & & & & \\ z^{l-1} & xz^{l-1} & & & & & & \end{array} \right\}.$$

Case (B) $y \notin \Gamma$**.** Since x^2 has the same weight as y, we have $x^2 \notin \Gamma$. Consider a unique monomial $\mathbf{m} = x^\alpha y^\beta z^\gamma$ in Γ of weight $2k-l+1$. From the fact that $\alpha \leq 1$ and $\gamma < l$, one can show that $\alpha = 1$ and $\gamma = 0$. The monomial $x^\alpha y^\beta z^\gamma$ is of weight $k-l$ and it is in Γ, so one can see that $\mathbf{m} = xy^{k-l}$. Furthermore, one can show that Γ contains y^{k-l+1}. Thus Γ is $\Gamma_l^{\triangle}$, i.e.

$$\Gamma = \left\{\begin{matrix} y^{k-l+1} & \\ y^{k-l} & xy^{k-l} \\ \cdots & \cdots \\ y & xy \\ 1 & x \\ z & xz \\ \cdots & \cdots \\ z^{l-1} & xz^{l-1} \end{matrix}\right\}.$$

Therefore the assertion is proved. $\square$

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